Set Cover

Given $S_1, S_2, \dots, S_n \subseteq S = \{1, 2, \dots, m\}$

Each S_j has a cost $c_j > 0$.

A cover $J \subseteq [n]$ satisfies

$$\bigcup_{j \in J} S_j = S. \quad \text{Cost } c(J) = \sum_{j \in J} c_j.$$

Problem: find cover that minimises cost.

$$x_j = \begin{cases} 1 & \text{use } S_j \\ 0 & \neg \text{use } S_j \end{cases} \qquad \text{Let } a_{ij} = \mathbb{1}_{i \in S_j} = \begin{cases} 1 & i \in S_j \\ 0 & i \notin S_j \end{cases}$$

minimise cost $= c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

subject to

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \geq 1 \qquad \underbrace{i \in \bigcup_{j \in J} S_j}_{\#(j : i \in S_j) \geq 1}, \ \forall i \in S$$

The a_{ij} are not variables to be computed.

They are data.

Further uses of integer variables

(1) Suppose variable x_j can take on of m values $p_1, p_2, \dots, p_m$

$$x_j = p_1\delta_1 + p_2\delta_2 + \dots + p_m\delta_m$$

$$\delta_1 + \delta_2 + \dots + \delta_m = 1$$

$$\delta_i = 0 \text{ or } 1, \ i = 1, 2, \dots, m$$

Usually all constraints must hold.

(11) $0 \leq x \leq M$

X is real

$\underset{\delta = 1}{0 \leq X \leq 1}$ OR $\underset{\delta = 0}{2 \leq X}$

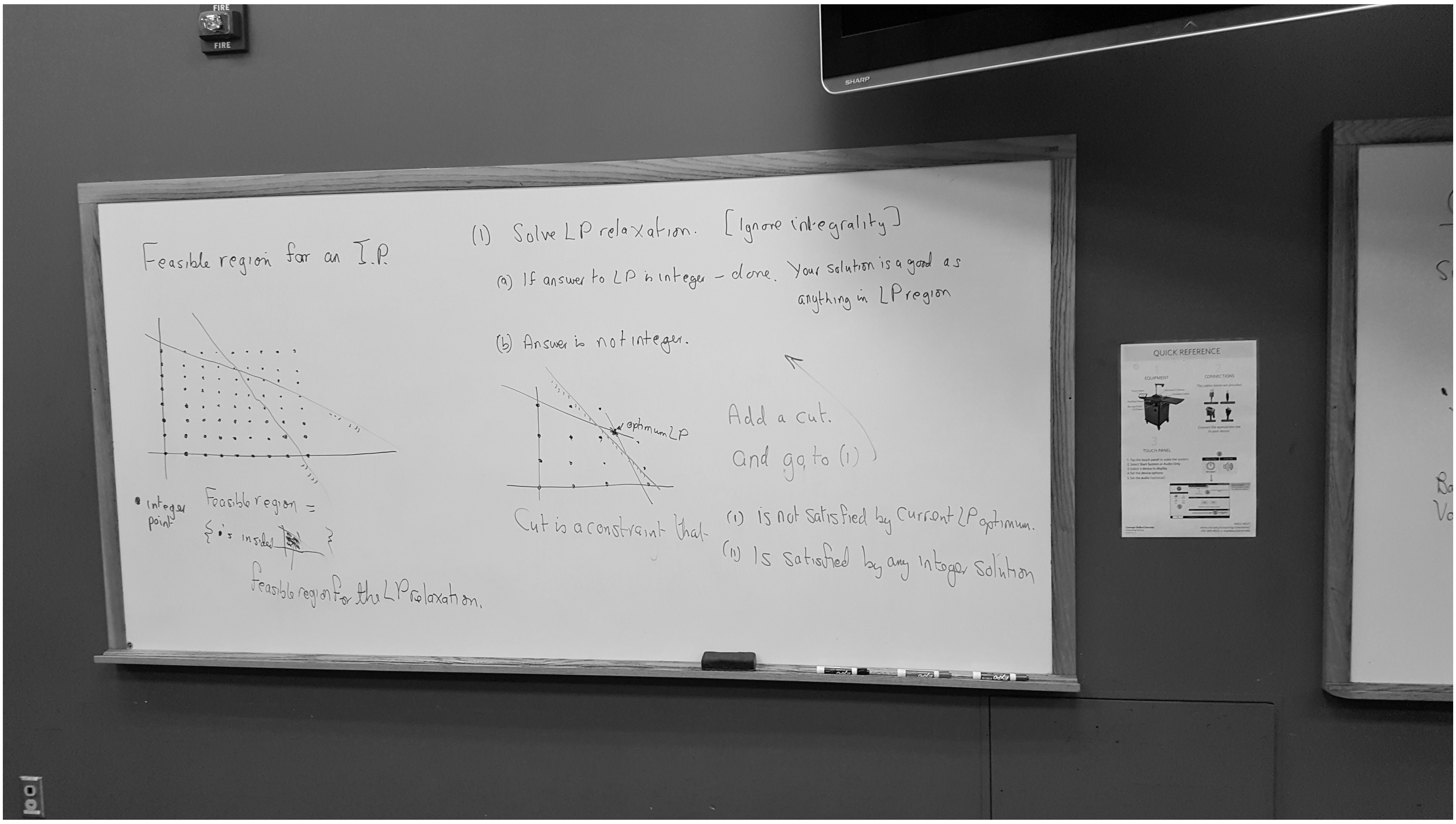
So we cannot have $X = 1\frac{1}{2}$

$$X \leq 1 + (M-1)(1-\delta)$$

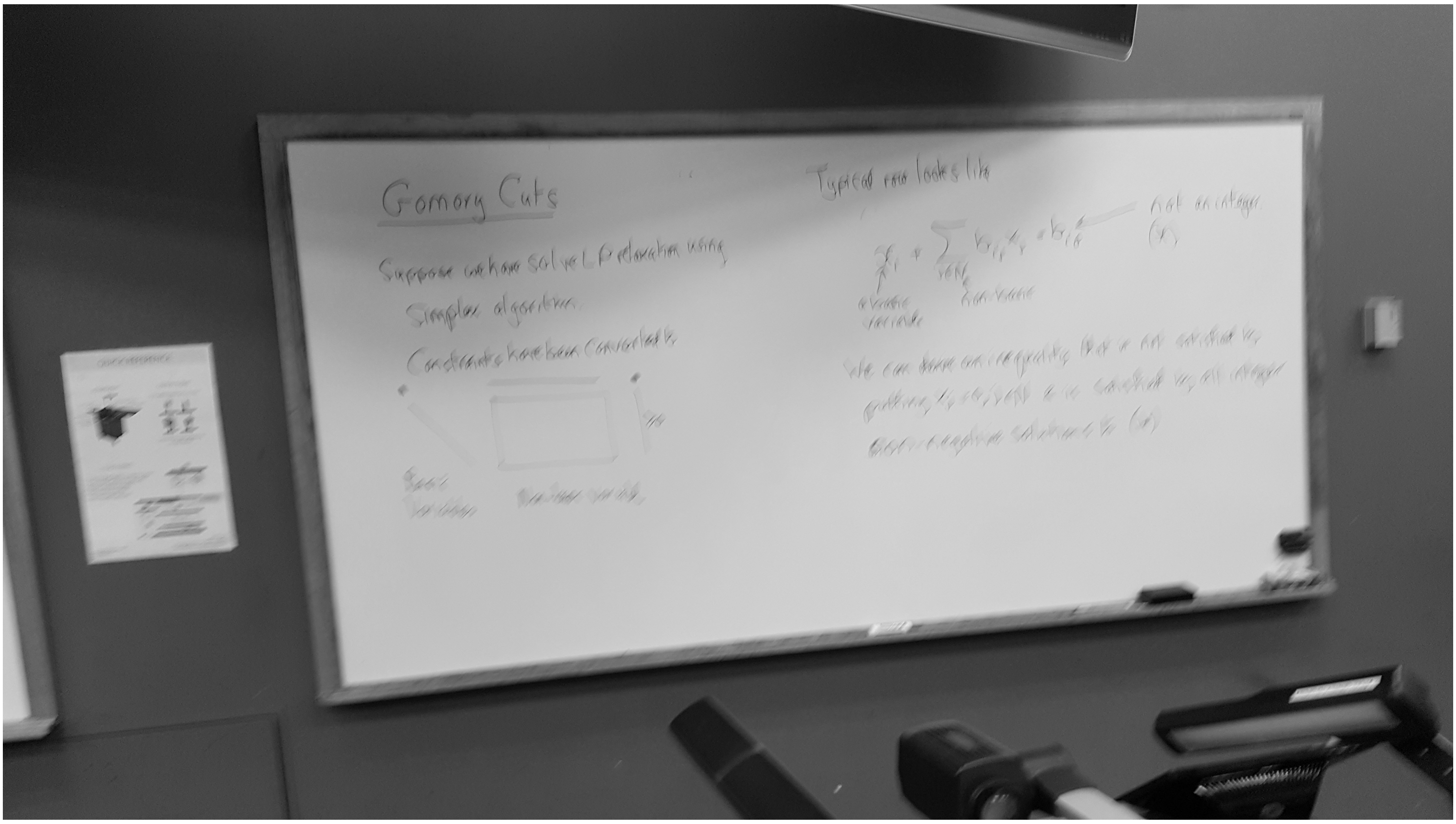
$$X \geq 2 - M\delta$$

$$\delta = 0, 1$$

$$X \geq 0$$



FIRE
FIRE
SHARP
Feasible region for an I.P.
• integer point
Feasible region =
{•'s inside }
feasible region for the LP relaxation.
(1) Solve LP relaxation. [Ignore integrality]
(a) If answer to LP is integer – done. Your solution is a good as anything in LP region
(b) Answer is not integer.
optimum LP
Add a cut.
and go to (1)
Cut is a constraint that
(i) is not satisfied by current LP optimum.
(ii) Is satisfied by any integer solution
QUICK REFERENCE


Gomory Cuts
Suppose we have solved LP relaxation using
Simplex algorithm
Constraints have been converted to
Typical row looks like
not an integer
(*)

Suppose that $x_1, x_2, \cdots, x_n \geq 0$ are integers and

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \qquad (*)$$

Let $a_i = \lfloor a_i \rfloor + f_i$, $i = 1, 2, \cdots, n$ and $b = \lfloor b \rfloor + f$ where $0 < f < 1$

Then

$$\underbrace{-\sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \lfloor b \rfloor}_{\text{integer}} = \underbrace{\sum_{j=1}^{n} f_j x_j - f}_{\geq -f}$$

So $\sum_{j=1}^{n} f_j x_j - f$ is an integer and > -1

$$\Longrightarrow \sum_{j=1}^{n} f_j x_j \geq f$$ for all non-negative integer solutions to $(*)$