Duality in LP

Value of primal feasible solutions

Minimise

Maximise

Value of dual feasible solutions

$$\begin{array}{cc} & v \in S_B \\ u \in S_A & \begin{bmatrix} \quad & & \\ & PAY[u,v] & \\ & & \quad \end{bmatrix} \end{array}$$

$P_A = \max_u \text{ROWMIN}(u)$

$P_B = \min_v \text{COLMAX}(v)$ $\qquad \underline{P_A \leq P_B}$

A can win P_A

B does not have to lose more than P_B

$P_A \leq P_B$ because

$\forall u,v \quad \text{ROWMIN}(u) \leq \text{PAY}(u,v) \leq \text{COLMAX}(v)$

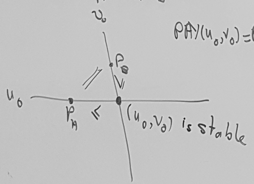
(u_0,v_0) is stable if for all u,v

$$\text{PAY}(u,v_0) \leq \text{PAY}(u_0,v_0) \leq \text{PAY}(u_0,v)$$

<u>Equilibrium</u>

<u>Claim</u>: $\exists$ a stable solution iff $P_A = P_B$

(I) Suppose $P_A = P_B$



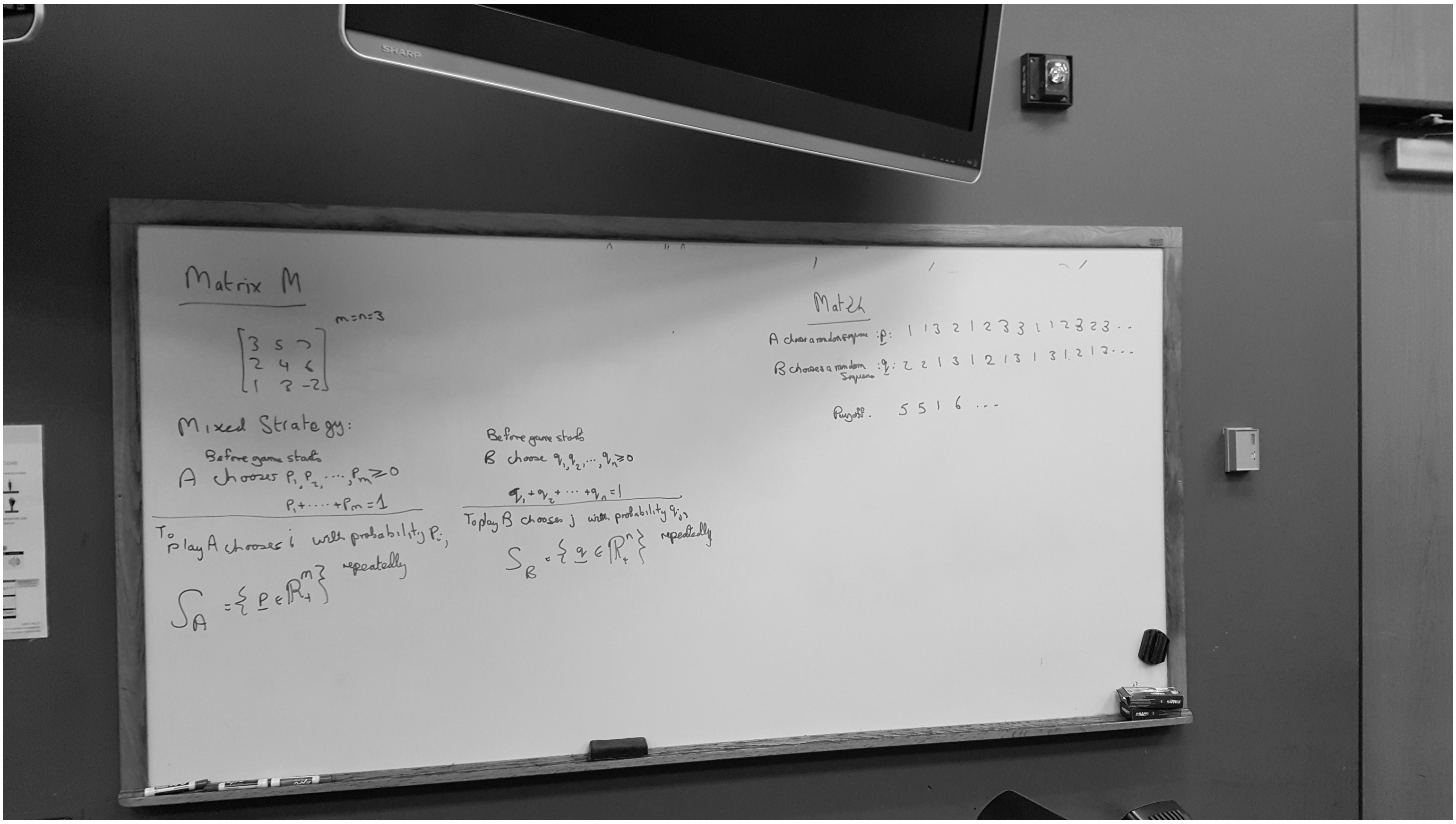
$\text{PAY}(u_0,v_0) = P_A = P_B$

(II) Suppose that (u_0,v_0) is stable.

$\geq \text{ROWMIN}(u_0) = \text{PAY}(u_0,v_0) = \text{COLMAX}(v_0) \geq \text{ROWMIN}(u)$

COLMAX(v)

$$\Rightarrow \begin{array}{l} \text{COLMAX}(v) \geq \text{PAY}(u_0,v_0) = P_B \\ \text{ROWMIN}(u) \leq \text{PAY}(u_0,v) = P_A \end{array}$$


Matrix M
m=n=3
3 5 7
2 4 6
1 3 -2
Mixed Strategy:
Before game starts
A chooses p1, p2, ..., pm ≥ 0
p1 + ... + pm = 1
To play A chooses i with probability pi, repeatedly
SA = { p ∈ R+m }
Before game starts
B choose q1, q2, ..., qn ≥ 0
q1 + q2 + ... + qn = 1
To play B chooses j with probability qj, repeatedly
SB = { q ∈ R+n }
Match
A chooses a random sequence : p : 1 1 3 2 1 2 3 3 1 1 2 3 2 3 . . .
B chooses a random sequence : q : 2 2 1 3 1 2 1 3 1 3 1 2 1 3 . . .
Payoff. 5 5 1 6 . . .

$$\mathrm{PAY}(\underline{p},\underline{q}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, p_i\, q_j$$

$$P_A = \max_{\underline{p}} \overbrace{\min_{\underline{q}} \mathrm{PAY}(\underline{p},\underline{q})}^{\mathrm{ROWMIN}(\underline{p})}$$

$$= \max_{\underline{p}} \min_{\underline{q}} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, p_i\, q_j$$

$$= \max_{\underline{p}} \min_{\underline{q}} \sum_{j=1}^{n} \underbrace{\left[\sum_{i=1}^{m} a_{ij} p_i\right]}_{c_j(\underline{p})} q_j$$

$$= \max_{\underline{p}} \min_{j} c_j(p)$$

Consider: minimise $3x_1 + 2x_2 + 4x_3$

s.t. $x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Similarly

minimise $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t. $x_1 + x_2 + \cdots + x_n = 1$

$x_j \geq 0, \forall j$

Solution value $\min\limits_{j=1,2,\ldots,n} c_j$

$$P_A = \max_{\underline{p}} \underbrace{\min_{j} c_j(\underline{p})}$$

$$= \max \xi : \xi \leq c_j(\underline{p}),\ j = 1, 2, \ldots, n$$

$$\max_{\underline{p}} \max_{\xi} : \xi \leq c_j(\underline{p}),\ \forall j$$

maximise ξ

s.t.

$$\xi \leq \sum_{i=1}^{m} a_{ij}\, p_i \qquad j = 1, 2, \ldots, n$$

$$1 = p_1 + p_2 + \cdots + p_m$$

$$p_1, p_2, \ldots, p_m \geq 0$$

QUICK REFERENCE

$$P_B = \min \eta$$

$$\eta \geq \sum_{j=1}^{n} a_{ij} q_j \quad , i=1,2,\cdots,m$$

$$1 = q_1 + q_2 + \cdots + q_n$$

$$q_1, q_2, \cdots q_n \geq 0$$

P_A & P_B solve a pair of dual linear programs and so $P_A = P_B$