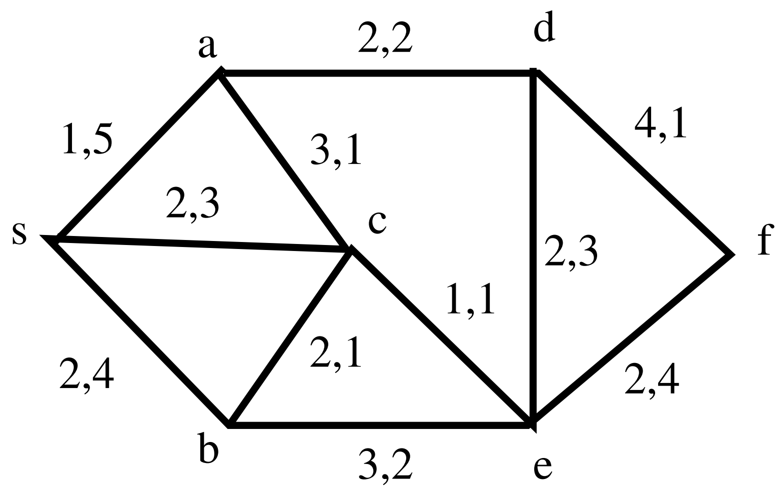
Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 2: Due Monday October 12.

1. Find a shortest path from s to all other nodes in the digraph below. Each edge (x, y) is labelled by a pair (a, b) and the length of the corresponding arc is $a + bt$ where t is the time the path reaches x. All arcs are directed lexicograhically e.g. (c, e) is directed from c to e.



2. Let $\mathcal{W}$ denote the set of walks in a directed graph D. If W_1 is a walk from a to b and W_2 is a walk from b to c then $W_1 + W_2$ is the walk from a to c obtained by following W_1 and then W_2.

   Let $\ell : \mathcal{W} \to \mathbb{R}$ be a real valued function defined on $\mathcal{W}$. Suppose that it has the following properties:

   (a) $\ell(C) \geq 0$ for any closed walk C. (A walk is closed if it begins and ends at the same vertex).

   (b) If W_1, W_1' are walks from a to b and W_2, W_2' are walks from b to c and $\ell(W_i') \geq \ell(W_i)$ for $i = 1, 2$ then $\ell(W_1' + W_2') \geq \ell(W_1 + W_2)$.

   Consider the following algorithm: n is the number of vertices in D.
   Initialise $W_{i,j} = (i, j)$ and $D_{i,j} = \ell(W_{i,j})$ for $i, j = 1, 2, \ldots, n$.

For $k = 1$ to n **Do**
    For $i = 1$ to n **Do**
        For $j = 1$ to n **Do**
        $D_{i,j} \leftarrow \min\{D_{i,j}, \ell(W_{i,k} + W_{k,j})\}$
        If $D_{i,j} = \ell(W_{i,k} + W_{k,j})$ **then** $W_{i,j} \leftarrow W_{i,k} + W_{k,j}$
        oD
    oD
oD

Prove that when the algorithm finishes,

$$D_{i,j} = \min\{\ell(P) : P \text{ is a path from } i \text{ to } j\}.$$

3. Suppose that the edges of a connected graph $G = (V, E = \{e_1, e_2, \ldots, e_m\}$ are given lengths $\ell(e_i), i = 1, 2, \ldots, m$ where $\ell(e_i) < \ell(e_{i+1}), 1 \leq i < m - 1$. Fot two spanning trees T_1, T_2 we say that $T_1 \prec T_2$ if there exists $r \leq n - 1$ ($n = |V|$) such that if $T_1 = \{e_{i_1}, e_{i_2}, \ldots, e_{i_{n-1}}\}$ and $T_2 = \{e_{j_1}, e_{j_2}, \ldots, e_{j_{n-1}}\}$ then $i_k = j_k$ for $1 \leq k < r$ and $i_r < j_r$.

   Prove that if T^* is the tree constructed by the Greedy Algorithm (Kruskal) and T is any other spanning tree, then $T^* \prec T$.