9/28/15

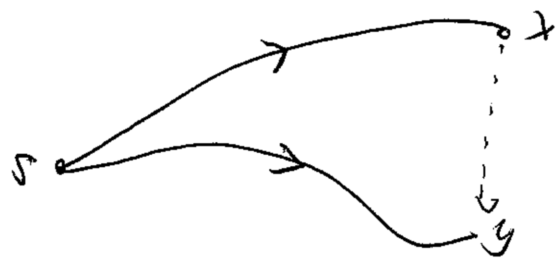
Shortest path with negative arc lengths.

$l(C) \geq 0$ for all directed cycles.

Optimality Condition

Start vertex s, Walk W_v from s to v for all $v \neq s$.

$$(*) \qquad l(W_x) + l(x,y) \geq l(W_y)$$



Clearly, if $(*)$ does not hold then $W_x + (x,y)$ is better than W_y and we are not optimal.

Suppose (*) holds and P is any walk from s to x. We have to show that $l(P) \geq l(W_x)$.

Let $d(y) = l(W_y)$ for all y.

So (*) says $d(a) + l(a,b) \geq d(b), \forall a,b$.

$P = (s = x_0, x_1, x_2, \ldots, x_m = x)$

$$l(x_{m-1}, x_m) \geq d(x_m) - d(x_{m-1})$$
$$l(x_{m-2}, x_{m-1}) \geq d(x_{m-1}) - d(x_{m-2})$$
$$\vdots$$
$$l(x_0, x_1) \geq d(x_1) - \cancel{d(x_0)}$$

Add $\quad l(P) \geq d(x_m)$.

Algorithm

$G = (V, E = \{e_1, e_2, \dots, e_m\})$

$e_i = (x_i, y_i)$

Initialise

$d(s) = 0$

$d(v) = \ell(s, v)$ $\quad W_v = s \to v$

Repeat

for $i = 1$ to m do

if $d(y_i) > d(x_i) + \ell(x_i, y_i)$

then put $d(y_i) = d(x_i) + \ell(x_i, y_i)$;

$W_{y_i} = W_{x_i} + (x_i, y_i)$

until (*) holds.

How long before (*) holds ?

Claim: Algorithm takes $\leq n-1$ iterations $(n = |V|)$.

O(mn) time

Proof

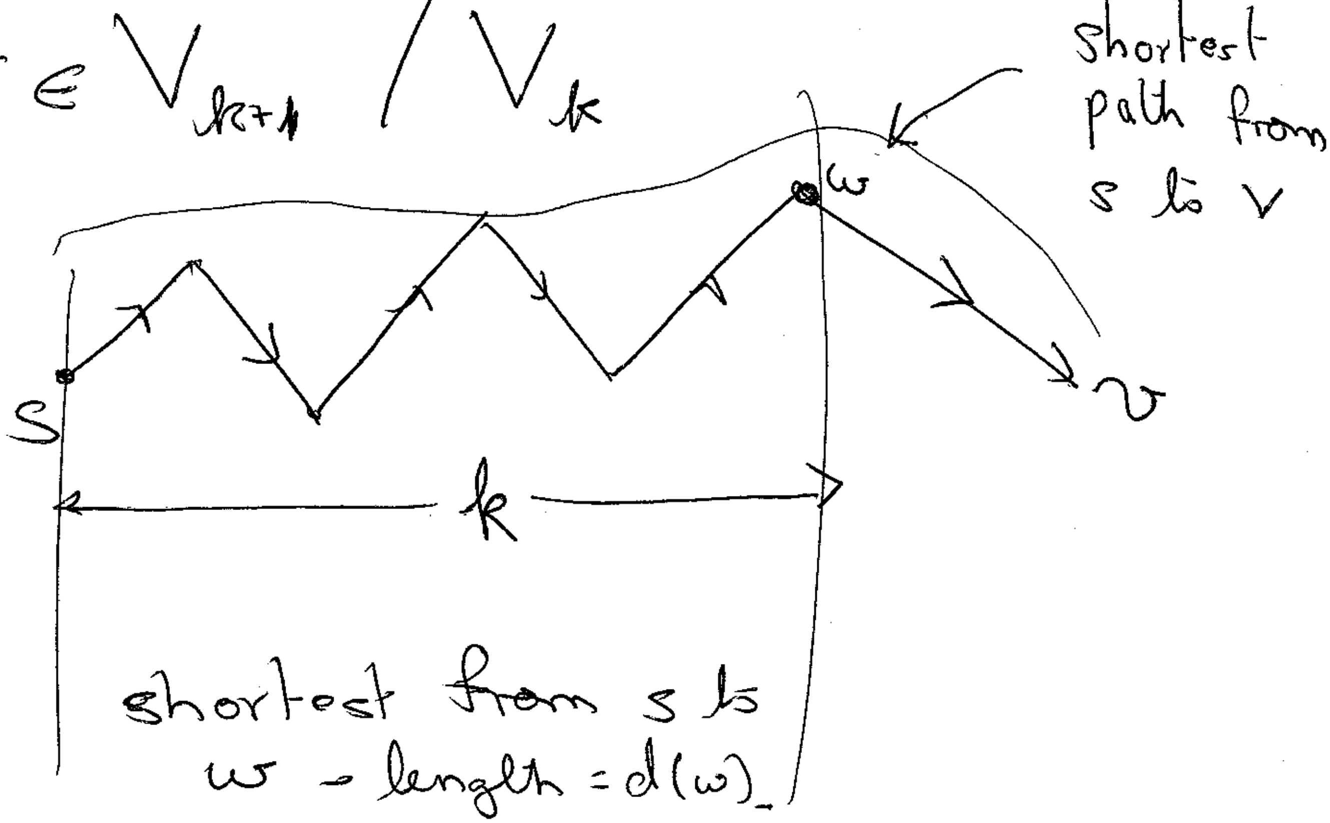
Let $V_k = \{ w : \exists$ a shortest path from s to v using $\leq k$ edges $\}$

Claim: after k iterations $d(v)$ is correct for $v \in V_k$

True for $k = 0$.

Assume true for some $k \geq 0$

$v \in V_{k+1} / V_k$



In the $k+1$'st round when we examine edge (w,v), $d(w)$ will be correct and after examining (w,v) we have $d(v) \leq \underbrace{d(w) + l(w,v)}_{\text{correct}}$,

All Pairs Shortest Paths

Start with a matrix l_{ij} = length of edge (i, j).

Initialise $d_{ij} = l_{ij}$ [$d_{ij} \to$ shortest distance eventually]

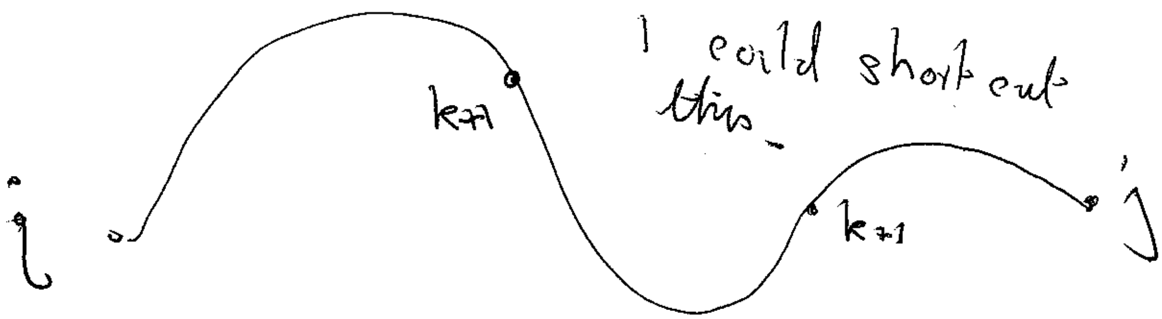
for $k = 1$ to n do
  for $i = 1$ to n do
    for $j = 1$ to n do
      $d_{ij} \leftarrow \min\{d_{ij}, d_{ik} + d_{kj}\}$
    od
  od
od

Claim: after completion of kth outer loop
d_{ij} = min length path i to j only using $1, 2, \ldots, k$ as interior nodes

Proof by induction on k.

$k=0$, easy.

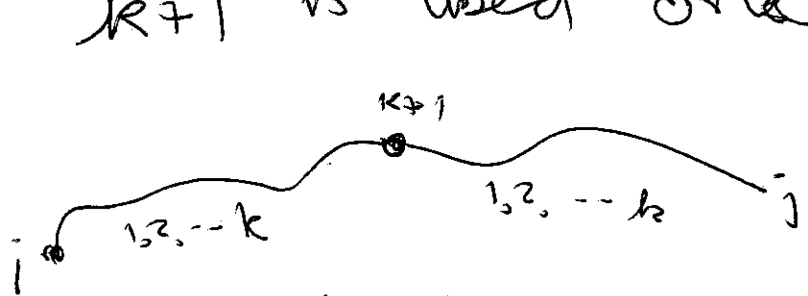
Assume true for k.



Shortest using $1, 2, \ldots, k+1$ in middle.

Case 1: $k+1$ is not used. d_{ij} is currently shortest.

Case 2: $k+1$ is used once



length $\geq d_{i,k+1} + d_{k+1,j}$