9/16/15

Choose π

evaluate π

Test for optimal

STOP

Improve π

Optimality Test

π minimizes $y_1, y_2, \cdots, y_n$ simultaneously iff

$$y_i = \min_{k=1}^{n} C(i,k) + \alpha y_k \quad *$$

Ex

$$\begin{bmatrix} 3 & 1 & 4 & 5 \\ 2 & 2 & 1 & 6 \\ 7 & 1 & 3 & 4 \\ 1 & 4 & 2 & 5 \end{bmatrix}$$

$\alpha = \frac{1}{2}$

i	1	2	3	4
π	$\cancel{4}$	4	1	4
y_i	10	11	12	10

Optimal?

y_1: $3 + \frac{1}{2} \times 10$, $1 + \frac{1}{2} \times 11$, $4 + \frac{1}{2} \times 12$, 10

y_2: $2 + \frac{1}{2} \times 10$, $2 + \frac{1}{2} \times 11$, $1 + \frac{1}{2} \times 12$, $6 + \frac{1}{2} \times 10$

y_3: $7 + \frac{1}{2} \times 10$, $1 + \frac{1}{2} \times 11$, $3 + \frac{1}{2} \times 12$, $4 + \frac{1}{2} \times 10$

y_4: $1 + \frac{1}{2} \times 10$, $4 + \frac{1}{2} \times 11$, $2 + \frac{1}{2} \times 12$, $5 + \frac{1}{2} \times 10$

Evaluation:

$$y_i = C(i, \pi(i)) + \alpha y_{\pi(i)}$$

This suggest that

i	$\cancel{1}$	2	3	4
$\pi(i)$	2	1	2	1

is a better policy.

$$\begin{bmatrix} 3 & 1 & 4 & 5 \\ 2 & 2 & 1 & 6 \\ 7 & 1 & 3 & 4 \\ 1 & 4 & 2 & 5 \end{bmatrix}$$

$$y_1 = 1 + \tfrac{1}{2} y_2 = 8/3$$
$$y_2 = 2 + \tfrac{1}{2} y_1 = 10/3$$
$$y_3 = 1 + \tfrac{1}{2} y_2 = 8/3$$
$$y_4 = 1 + \tfrac{1}{2} y_1 = 7/3$$

Suppose π passes optimality test.

Let $\hat{\pi}$ be any other policy and let its evaluation be $\hat{y}_1, \hat{y}_2, \ldots \hat{y}_n$

$$\hat{y}_i = c(i, \hat{\pi}(i)) + \alpha \hat{y}_{\hat{\pi}(i)} \quad \forall i$$

$$y_i \leq c(i, \hat{\pi}(i)) + \alpha y_{\hat{\pi}(i)}$$

$$\underbrace{\hat{y}_i - y_i}_{z_i} \geq \alpha\left(\hat{y}_{\hat{\pi}(i)} - y_{\hat{\pi}(i)}\right)$$

$$z_i \geq \alpha z_{\hat{\pi}(i)}$$

$$\geq \alpha^2 z_{\hat{\pi}^2(i)}$$

$$\geq \ldots \geq \alpha^k z_{\hat{\pi}^k(i)} \to 0 \implies \underline{z_i \geq 0}$$

Now suppose that π fails the optimality test

Choose $\hat{\pi}(i)$ to minimise $c(i,k) + \alpha y_k$.

$$y_i \geq c(i, \hat{\pi}(i)) + \alpha y_{\hat{\pi}(i)} \qquad > \text{ in some places}$$

$$\hat{y}_i = c(i, \hat{\pi}(i)) + \alpha \hat{y}_{\hat{\pi}(i)}$$

$$\cancel{\hat{y}_i = c(i, \hat{\pi}(i)) + \alpha \hat{y}_{\hat{\pi}(i)}}$$

Put $z_i = y_i - \hat{y}_i$ and argue that

$$z_i \geq \alpha^k z_{\hat{\pi}^{(k)}(i)} \longrightarrow 0$$

& $z_i > 0$ in some places

Linear Programming Solution

Solve $\quad y_i = \min_k \left[c(i,k) + \alpha y_k \right]$

$$y_i \leq c(i,k) + \alpha y_k \quad \forall i,k$$

maximise $\quad y_1 + y_2 + \cdots + y_n \quad$ s.t.