10\14\15

Matrix A $m \times n$

$S_A = \{(p_1, p_2, \ldots, p_m) : p_i \geq 0 \ \forall i \ \& \ p_1 + \cdots + p_m = 1\}$

$\{$Randomized strategies for $A\}$

$S_B = \{(q_1, \ldots, q_n) : q_j \geq 0 \ \forall j \ \& \ q_1 + \cdots + q_n = 1\}$

Payoff: $PAY(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \, p_i \, q_j$

In this situation $P_A = P_B$

$$P_A = \max_{p \in S_A} \min_{q \in S_B} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, p_i\, q_j$$

$$= \max_{p \in S_A} \left[\min_{q \in S_B} \sum_{j=1}^{n} \Big(c_j(p) \Big) q_j \right]$$

$$\sum_{i=1}^{m} c_{ij}\, p_i$$

Consider minimise $3q_1 + 4q_2 + 2q_3$

s.t.

$$q_1 + q_2 + q_3 = 1$$

$$q_1, q_2, q_3 \geq 0.$$

$$P_A = \max_{p \in S_A} \min_{j = 1 \cdots m} c_j(p)$$

— Linear Program

$$\rho_A = \max_{p \in S_A} \quad \max \ \xi$$

$$\xi \leq c_1(p)$$
$$\xi \leq c_2(p)$$
$$\vdots$$
$$\xi \leq c_n(p)$$

$$\rho_A = \max \ \xi$$

$$-\xi + \sum_{i=1}^{m} c_{i1} \, p_i \geq 0$$

$$-\xi + \sum_{i=1}^{m} c_{i2} \, p_i \geq 0$$

$$\vdots$$

$$-\xi + \sum_{i=1}^{m} c_{in} \, p_i \geq 0$$

$$p_1 + p_2 + \cdots + p_m = 1$$

$$p_1, p_2, \ldots, p_m \geq 0.$$

$$P_B = \min_{q \in S_B} \max_{p \in S_A} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

$$= \min_{q \in S_B} \left[\max_{p \in S_A} \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} q_j \right)}_{d_j(q)} p_i \right.$$

$$= \min_{q \in S_B} \max \; d_1(q), d_2(q), \ldots, d_m(q)$$

$$\min \quad \eta$$

$$\eta \geq d_1(q)$$

$$\eta \geq d_2(q)$$

$$\vdots$$

$$\eta \geq d_m(q)$$

$$q_1 + \cdots + q_n = 1$$

$$q_1, \ldots, q_n \geq 0$$

LP

$$P_A$$

max ξ

$$-\xi + \sum_{i=1}^{m} a_{i1} p_i \geq 0$$

$$\vdots$$

$$-\xi + \sum_{i=1}^{m} a_{in} p_i \geq 0$$

$$p_1 + \cdots + p_m = 1$$

$$p_1, \ldots, p_m \geq 0$$

$$P_B$$

min η

$$-\eta + \sum_{j=1}^{n} a_{1j} q_j \leq 0$$

$$\vdots$$

$$-\eta + \sum_{j=1}^{n} a_{mj} q_j \leq 0$$

$$q_1 + \cdots + q_n = 1$$

$$q_1, \ldots, q_n \geq 0$$

These two LP's are dual to each other - check.

$$\Rightarrow \quad P_A = P_B$$

Dominance

B

$$A \quad \begin{bmatrix} 4 & 3 & 6 & 5 & 4 \\ 3 & 2 & 5 & 4 & 3 \\ 5 & 1 & 7 & 2 & 2 \end{bmatrix}$$

2 is dominated by 1.

1 < 2   3 < 2

dominated by

A play 1, & B play 2

Symmetric Game

A is $n \times n$ and $A^T = -A \Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i p_j = 0$

$a_{ii} = 0$

$a_{ij} p_i p_j + a_{ji} p_j p_i$

$$PAY(p, q) \quad \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{bmatrix}$$

$\infty \times \infty$ matrix

$P_A \leq 0$

$P_B \geq 0$

$P_A = P_B = 0$

Both players play same

Example R, P, S

$$\begin{array}{c|ccc} & R & P & S \\ \hline R & & & \\ P & & & \\ S & & & \end{array}$$

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{matrix} R & P & S \end{matrix} \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

Solution $p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ — Check LP's

$q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Non-Singular Games

A is non-singular

$$v = \frac{1}{\underline{1}^T A^{-1} \underline{1}} \quad \& \quad p = v A^{-T} \underline{1}$$

$$q = v A^{-1} \underline{1}$$

If $p, q \geq 0$ then p, q solve the game.

$p_A := v$?

Put $p = v A^{-T} \mathbb{1} \geq 0$ $\qquad \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$$\mathbb{1}^T p = v\, \mathbb{1}^T A^{-T} \mathbb{1} = 1$$

$$-\xi \mathbb{1} + A^T p \geq 0 \quad \leftarrow \text{Remaining constraints}$$

? $\xi = v$

$p = v A^{-T} \mathbb{1}$ satisfies ?

$$-v + v A^T A^{-T} \mathbb{1} = 0$$