10/12/15

Two Person-Zero Sum Games

A game is defined by a matrix.

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 6 & 4 & -3 & 4 \\ 2 & -1 & 3 & 1 \\ 4 & 2 & -2 & 4 \\ 0 & 3 & 4 & 5 \end{bmatrix} \end{array}$$

Two Players: A & B
row player, column player

A chooses row i
B chooses col. j
— without each others knowledge.

Then B pays A an amount a_{ij}

How to play?

Rock, Paper, Scissors

$$
\begin{array}{c c}
 & \begin{array}{ccc} R & P & S \end{array} \\
\begin{array}{c} R \\ P \\ S \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{array}
$$

We play forever.

Receiver or Batter or Goalkeeper

Tennis Server or Pitcher or Soccer Penalty kicker

$$
\left[\; a_{ij} \text{ is the expected outcome} \; \right]
$$

What is a solution?

Stable Solution

A has a set of strategies S_A & $PAY[u,v]$

B has a set of strategies S_B = payoff [expected]

What is the problem?

$$\begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

(u_0, v_0) is Stable if neither player has an incentive to change.

$$PAY(u, v_0) \leq PAY(u_0, v_0) \leq PAY(u_0, v)$$

$\forall u$ $\qquad \forall v$

Player A might try to maximize guaranteed winnings

A should choose u to maximise ROWMIN(u)

$$P_A = \max_u \text{ ROWMIN}(u)$$

A can always get this much

$$P_B = \min_v \text{ COLMAX}(v)$$

B can always lose at most this.

Thm

(I) $P_A \leq P_B$

(II) $P_A = P_B$ iff there is a stable solution.

$$* \begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$P_A = 1 \qquad P_B = 3$

Mixed Strategy

A chooses $p_1, p_2, \dots, p_m \geq 0 \quad p_1 + \dots + p_m = 1$

B choose $q_1, q_2, \dots, q_n \geq 0 \quad q_1 + \dots + q_n = 1$

Then A plays i with probability p_i

B plays j with probability q_j

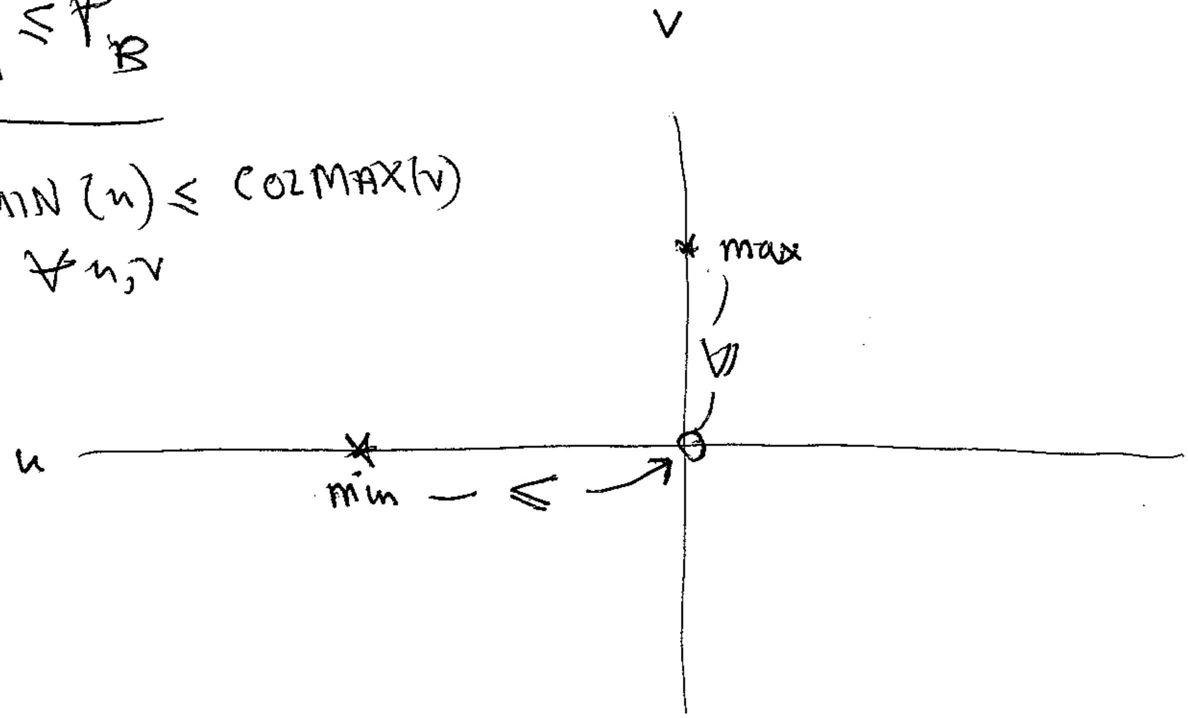
Expected payoff

$$PAY[p, q] = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

Thm $\exists$ a stable solution p_0, q_0 [Duality in LP

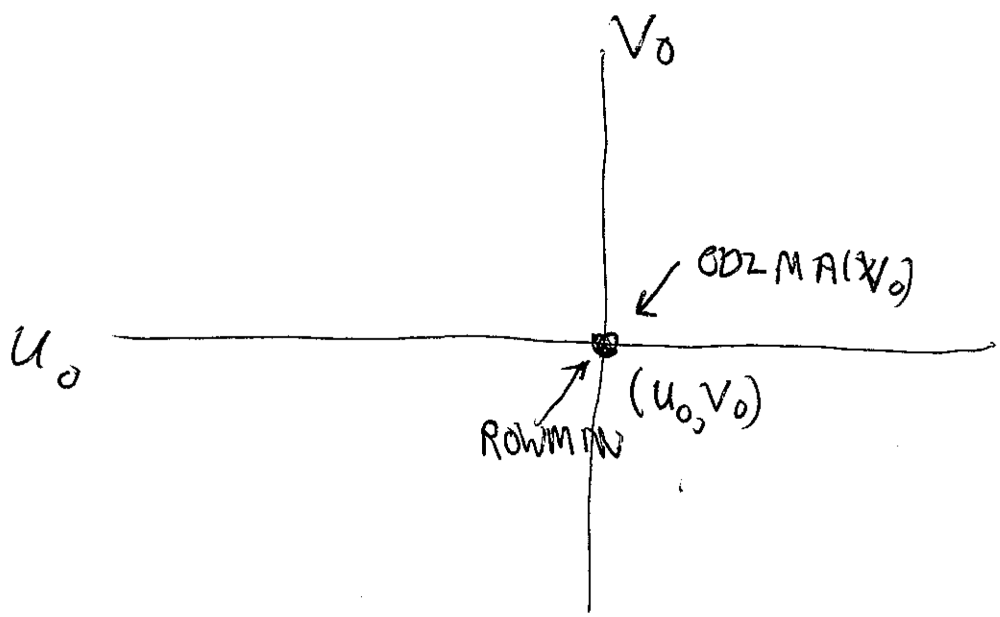
$\rho_A \leq \rho_B$

$\mathrm{ROWMIN}(u) \leq \mathrm{COLMAX}(v)$
$\forall u, v$



$\rho_A = \rho_B$ iff there is a stable solution.

① Suppose (u_0, v_0) is stable



$$\mathrm{COLMAX}(v) \geq \underbrace{\mathrm{ROWMIN}(u_0)}_{\rho_B} = \underbrace{\mathrm{COLMA}(V_0)}_{\rho_A} \geq \mathrm{ROWMIN}(u) \quad \forall u$$

(11) Suppose $P_A = \text{ROWMIN}(u_0)$

$$P_B = \text{COLMAX}(v_0)$$

$$P_A = P_B$$

$$\cancel{\text{ROWMIN}(u_0) = \text{PAY}(u_0, v_0) \leq \text{PAY}(u_0, v)}$$

$$\cancel{P_A}$$

$P_B = \text{COLMAX}(v_0)$

$\geq$

$P_A = \text{ROWMIN}(u_0) \leq \text{PAY}(u_0, v_0)$

So $\text{PAY}(u_0, v_0) =$

So these 3 values are equal $[P_A = P_B]$