Shortest Path Problem.

Digraph = 
(N, A)

Nodes  Arrows

Path: no vertex is repeated
Walk: any sequence of arrows connected.
Problem: Find minimum length path $S\rightarrow t$. 

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Negative Cycle

If $D$ has a cycle $C$ with $l(C) < 0$ then no guarantee of an efficient algorithm.

Assume: No negative cycles.

Shortest path from $s \rightarrow t$ is also the shortest walk.

\[ \text{remove cycle, reduce the cost.} \]
We can look for a shortest walk.

Suppose $P_j : j \in N$ is a collection of paths where $P_j$ is a path from $s$ to $j$.

$$d_j = \ell(P_j).$$

Then

These are all shortest paths iff

$$d_j \leq d_i + \ell(ij), \quad \forall (i,j) \in A.$$
Proof

Only if: \( d_j > d_i + l(i, j) \)

lower is a shorter walk from \( s \) to \( j \)

If:

Any path \( s \to i_1 \to \ldots \to i_p \to j \)

\( l(P) \geq d(j) \) \( \iff \)

\( d(i_p) - d(i_{p-1}) \leq l(i_p, i_{p-1}) \)

\( d(i_{p-1}) - d(i_p) \leq l(i_p, i_{p-1}) \)

\( d(i, j) - d(s) \leq d(s, i) \)
Ford's Algorithm

\[ A = \{ u_1, u_2, \ldots, u_m \} \]

\[ u_i = \{ x_i, y_i \} \]

\[ d(s) = 0; \quad d(v) = \infty \quad \text{for } v \neq s; \]

Repeat Until Optimality Condition Holds

\[
\text{for } i = 1, 2, \ldots, m
\]

\[
\text{if } d(y_i) > d(x_i) + b(x_i, y_i)
\]

\[
\text{then } d(y_i) = d(x_i) + b(x_i, y_i); \quad \pi(y_i) = x_i
\]

Stop if you go through loop without a change.
Ford's algorithm terminates with optimum paths.

\( d(s) \) is always correct.

\( S(k) = \{ v : \text{shortest path with } k \text{ edge} \} \)

\( d(v) \) is correct after \( k \) round: Induction