Birth and death chain

Let \( \omega_h = \prod_{i=1}^{h} \frac{p_{i-1}}{q_i} \) with \( \omega_0 = 1 \).

For \( h \geq 1 \),
\[
\begin{align*}
p_{k-1} \omega_{k-1} &= q_k \omega_k \\
p[k \to k-1] &= p[k \to k-1]\n\end{align*}
\]

\( \omega_0, \omega_1, \ldots, \omega_h \) satisfy detailed balance.
So

(i) Chain is reversible

(ii) \( \Pi_k \propto \frac{\omega_k}{w_0 + \ldots + w_n} \) proportional

Fix \( l \in \{0, 1, \ldots, n \} \) and restrict chain

\( L = \{0, 1, 2, \ldots, l \} \)

Same moves as before, except that

\[ P_{e, e'}, R \rightarrow 0, c_{e'}, P_e + R \]

\( \frac{\omega_k}{\Pi_k} = \text{steady state for new chain} = \frac{\omega_k}{w_0 + \ldots + w_n} \) check balance
$E^i$: expectation w.r.t. this chain.

Hitting Times

$E_l(T_l^+)=1+q_le_{l-1}(T_l)$

$=E_l(T_l)$

$1/\pi_l$

$=\frac{1}{\omega_0+\omega_1+...+\omega_l}$

$\omega_l$

$E_{l-1}(T_l)=\frac{1}{q_le_{j=0}^{l-1}}\sum_{j=0}^{l-1} \omega_j$
\[ E_a(P_b) = \sum_{b=a+1}^{b} E_{b-1}(P_b) \]

\[ a < b \]

Special Case: \( (P_c, q_c, r_c) = (P, q, r) \)

\[ k \neq 0, n \]

\[ p \neq q: \omega_{bc} = \left( \frac{p}{q} \right)^k, \quad 0 < k < n \]

\[ E_{l-1}(P_b) = \frac{1}{q(l/q)^l} \sum_{j=0}^{l-1} (p/q)^j = \frac{1}{p-q} \left[ 1 - \left( \frac{q}{p} \right)^l \right] \]

\[ p = q: \omega_{bc} = 1 \quad E_{l-1}(P_b) = \frac{l}{p} \]
Random Walks on Groups

$G$ is a finite group.

$\mu$ is a measure on $G$.

$\sum_{h \in G} \mu(h) = 1$

Cube: $G = \mathbb{Z}_n$ for $e_1, e_2, \ldots, e_n$

$\mu = \frac{1}{n}, \frac{1}{n}, \frac{1}{n}

Cycle $G = \mathbb{Z}_n$

$\mu(1) = \frac{1}{2}$
$\mu(-1) = \frac{1}{2}$
Prop 2.12

$P$ is transition matrix of random walk on group $G$. Then $U = \text{uniform distribution}$ is stationary.

Proof

Detailed balance fails

$$U(g) P(g, h g) \neq U(h g) P(h g, g)$$

$$\Rightarrow \quad m(h) \frac{m(h^{-1})}{m(h)}$$

(i) $m(h) = m(h^{-1}) \Rightarrow U(g) \text{ is stationary}$

(ii) Chain is reversible.

Otherwise, chain is not reversible.
\[
\sum_{h \in G} U(h) P(h, g) = \frac{1}{|G|} \sum_{k \in G} P(k^{-1} g, g)
\]

\[
U(g) = \frac{1}{|G|} \sum_{h \in G} \mu(h)
\]

2.6.1
Generating set, irreducibility, reversibility.
If \( H \leq G \) then \( \langle H \rangle \) = subgroup generated by \( H \)

\( H \) is a set of generators if \( \langle H \rangle = G \)
Prop 2.13

Random walk is irreducible ⇔ \( \langle \sum_{h_i} \mu(h_i) > 0 \rangle = 0 \).

Proof:

\( \exists k : P^k(\text{id}, a) > 0 \Rightarrow \exists a_1, a_2, \ldots, a_n \in H \) such that \( a = a_n a_{n-1} \ldots a_1 \).

Suppose \( a, b \in G \). Can write \( ba^{-1} = s_1 s_2 \ldots s_i \).

\( P'(a, b) = P(a, s, a) P(s, a, s, s, a) \ldots s_i \in H \).
Walk is symmetric if $\mu(g) = \mu(g^{-1})$ for $g$.

Revers =de
2.7 Random walks on $\mathbb{Z}$ and Reflection principle.

Thm 2.12

$$P_k(T_0 > r) < \frac{12k}{\sqrt{r}}$$
Lemma (Reflection Principle)

\[ P_k( T_0 < r, X_r = i) = P_k( X_r = -i) \]

\[ P_k( T_0 < r, X_r > 0) = P_r( X_r < 0) \]

**Proof**

\( (a) \Rightarrow (b) \): sum over \( i > 0 \).

\[ P_k( T_0 = s, X_r = i) = P_k( T_0 = s) P_0( X_{r-s} = i) \]  
\( s < r \) and \( i > 0 \)

\[ = P_k( T_0 = s) P_0( X_{r-s} = -i) \]  \( \text{ll reflection} \)

\( \text{Sum over } s < r : P_k( T_0 < r, X_r = i) = P_k( X_r = -i) \text{ for } i > 0 \)