Vertex Colourings

We assume in this chapter that $G$ is simple.

A $k$ - colouring of (the vertices of) $G$ is a mapping

$$c : V \rightarrow \{1, 2, \ldots , k\}.$$  

c$(v)$ is the colour of vertex $v$.

$K_i = \{v \in V : c(v) = i\}$ is the set of vertices with colour $i$.

c is proper if $K_1, K_2, \ldots , K_k$ are independent sets i.e. adjacent vertices $v, w$ have $c(v) \neq c(w)$. 

1
$G$ is $k$-colourable if it has a proper $k$-colouring.  
A graph is $k$-colourable iff it is $k$-partite.  
The Chromatic Number

$$\chi(G) = \min\{k : G \text{ is } k\text{-colourable}\}.$$  

**Lemma 1**  

$$\chi(G) \geq \max\{\text{cl}(G), \nu/\alpha(G)\}$$  
where $\text{cl}(G)$ is the size of the largest clique in $G$.  

**Proof**  
If $C$ is a clique of $G$ then every vertex of $C$ must have a different colour in a proper colouring of $G$.  
If $K_1, K_2, \ldots, K_k$ defines a proper $k$-colouring then

$$\nu = \sum_{i=1}^{k} |K_i| \leq k\alpha(G).$$
Greedy Colouring Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$ and $V_i = \{v_1, v_2, \ldots, v_i\}$ for $i = 1, 2, \ldots, n$.

begin

for $i = 1$ to $n$ do

begin

$c(v_i) := \min\{j : \exists w \in N_G(v_i) \cap V_{i-1} \text{ with } c(w) = j\}$

end

end
Theorem 1

\[ \chi(G) \leq \Delta(G) + 1. \]

The Greedy Colouring algorithm produces a proper \( k \)-colouring for some \( k \leq \Delta + 1 \) where

\[ k \leq 1 + \max_i |N_G(v_i) \cap V_{i-1}|. \]  

(1)

(a) The colouring is proper: Suppose \( v_r v_s \in E \) and \( r < s \). \( c(v_r) \neq c(v_s) \) since \( c(v_s) \) is the lowest numbered colour that is not used by a neighbour of \( v_s \) in \( \{v_1, v_2, \ldots, v_{s-1}\} \),

(b) At most \( \Delta + 1 \) colours are used: \( |N_G(v_i)| \leq \Delta \) and so the minimum above is never more than \( \Delta + 1 \).

If \( G \) is a complete graph or an odd cycle then \( \chi(G') = \Delta + 1. \)
Colouring Number

Let

\[ \delta^*(G) = \max_{S \subseteq V} \delta(G[S]) \]

(the maximum over the vertex induced subgraphs of their minimum degrees.)

\[ \delta(G) = 2 \text{ and } \delta^*(G) = 3. \]
Theorem 2

\[ \chi(G') \leq \delta^*(G) + 1. \]

Proof  Let \( V = \{v_1, v_2, \ldots, v_n\} \) where

\( v_i \) is a minimum degree vertex of \( G[V_i] \).

\[
|N_G(v_i) \cap V_{i-1}| = \delta(G[V_i]) \leq \delta^*.
\]

The theorem follows from (1). \( \square \)
Brook’s Theorem

Theorem 3 If $G$ is a connected graph which is not a complete graph or an odd cycle then $\chi(G) \leq \Delta(G)$.

Proof We shall prove this by induction on the number of vertices in $G$.

Assume that $G$ is connected but not a complete graph or an odd cycle.

If $G$ has a cutpoint $v$ let $G - v$ have components $C_1, C_2, \ldots, C_p$ and let $W_i = C_i + v$ for $i = 1, 2, \ldots, p$. Let $k_i = \chi(G[W_i])$ and properly $k_i$-colour the vertices of each $W_i$ so that $v$ has colour 1 in each.
This induces a proper $k$-colouring of $G$ where $k = \max\{k_1, k_2, \ldots, k_p\}$.

We argue that $k \leq \Delta$. If say $k_1 = \Delta + 1$ then (by induction) either $W_1$ is an odd cycle or a complete graph on $k_1$ vertices.
If $W_1$ is an odd cycle then $k_1 = 3$ and $\Delta = 2$ but now $d_G(v) \geq 3$ — contradiction.

If $W_1$ is a complete graph on $k_1$ vertices then $\Delta \geq d_G(v) \geq k_1$ — contradiction.

Suppose next that $G$ contains a vertex $v$ with $d_G(v) \leq \Delta - 1$. Let $H = G - v$.

If $H$ is an odd cycle then $\Delta(G) = 3$. We can 3-colour $H$ and then colour $v$ with a colour not used by one of its $\leq 2$ neighbours. Thus $\chi(G) = 3$ as required.
If $H$ is a $k$-clique then $\Delta(G) = k$. We $k$-colour $H$ and extend the colouring to $v$ as $v$ has less than $k$ neighbours in $H$.

If $H$ is is neither a clique or an odd cycle then we can $\Delta$-colour it. We can extend this colouring to $v$ by using one of the colours not used so far in $N_G(v)$.

We can therefore assume that $G$ is $\Delta$-regular and 2-connected with $\Delta \geq 3$. 
We now consider 2-vertex cutsets. Suppose first that $G$ contains vertices $u, v$ such that $uv \in E$ and $u$ is a cut point of $H = G - v$.

Let $C_1, C_2, \ldots, C_k$ be the components of $H - v$. Each $C_i$ contains at least one neighbour $x_i$ of $v$, else $u$ is a cutpoint of $G$.

Take a $\Delta$-colouring of $H$. Assume first that all neighbours of $u$ have different colours. Interchange colours $c_1, c_2$ of $x_1, x_2$ within $C_2$ only.
Because $u$ does not have colour $c_1$ or $c_2$ and $C_1$ has no neighbours other than $u$ we see that this yields a new proper colouring of $H$, but now $x_1$ and $x_2$ have the same colour $c_1$.

Thus we can assume that we have a $\Delta$-colouring of $H$ in which 2 neighbours of $v$ have the same colour. This colouring can be extended to $v$ since fewer than $\Delta$ colours are being used by neighbours of $v$. 
Suppose then that there are no two neighbours which form a 2-vertex cut set. We prove the existence of vertices $a, b, c$ such that

$$ab, ac \in E \text{ and } bc \notin E \text{ and } G - \{b, c\} \text{ is connected.}$$

(2)

Choose $y \in V$ and let $x$ be at distance 2 from $x$. $y$ cannot be a neighbour of every other vertex else $G$ is $(\Delta + 1)$-clique. Let $x$ be the middle vertex of a path from $x$ to $y$ of length 2. Then $xy, xz \in E$ and $yz \notin E$.

If $G - \{yz\}$ is connected then let $a, b, c = x, y, z$. 
Otherwise let $G - \{yz\}$ have components $C_1, C_2, \ldots, C_k$. $y$ has a neighbour $\alpha \neq x$ in $C_1$ else $x$ is of degree 2 or is a neighbour of $z$ which is a cutpoint of $G - z$. Similarly, $y$ has a neighbour $\beta \neq x$ in $C_2$.

We claim that $H = G - \{\alpha, \beta\}$ is connected and so we can take $a, b, c = y, \alpha, \beta$. 
Suppose $C_2 - \beta$ has components $D_1, D_2, \ldots$. Then $z$ is adjacent to $D_1$ else $\beta$ is a cutpoint of $G - y$. Similarly, $z$ is adjacent to all components of $C_1 - \alpha$ and $C_2 - \beta$. Now $H$ contains the path $x, y, z$ and every other component $C_3, \ldots, C_k$ is connected to $y, z$ and so $H$ is connected.
Suppose that (2) holds. We run the Greedy colouring algorithm with

\[ v_1 = b, v_2 = c, v_3, \ldots, v_{n-1}, v_n = a \]

The sequence \( v_3, \ldots, v_{n-1}, v_n \) is obtained by doing BFS from \( a \) in \( G - \{b, c\} \).

The important thing is that for \( 3 \leq i \leq n - 1 \)

\[ \exists j > i \text{ such that } v_j \text{ is a neighbour of } v_i. \]  \( (3) \)
Greedy uses at most $\Delta$ colours.

$v_1$ and $v_2$ both get colour 1.

For $3 \leq i \leq n - 1$, (3) implies that at most $\Delta - 1$ of $v_i$’s neighbours have already been coloured when we come to colour $v - i$.

Finally, $v_n = a$ has at least 2 neighbours, $b, c$ using the same colour and so at most $\Delta - 1$ colours have been used so far in $a$’s neighbourhood. \qed
Chromatic Polynomial

\( \pi_k(G) \) is the number of distinct proper \( k \)-colourings of \( G \).

\[
\pi_k = k(k-1)(k-2)
\]

**Theorem 4** Let \( e = uv \) be an edge of \( G \). Then

\[
\pi_k(G) = \pi_k(G - e) - \pi_k(G \cdot e).
\]

**Proof** \( \pi_k(G) \) = the number of \( k \)-colourings of \( G - e \) in which \( u, v \) have different colours.

\( \pi_k(G \cdot e) \) = the number of \( k \)-colourings of \( G - e \) in which \( u, v \) have the same colour. \( \square \)
Theorem 5 $\pi_k(G)$ is a polynomial of degree $\nu$ in $k$ with integer coefficients, leading term $k^\nu$ and constant term zero. The coefficients alternate in sign.

Proof By induction on $|E|$. If $E = \emptyset$ then $\pi_k(G) = k^\nu$.

Assume true for all graphs with $< m$ edges and let $G$ be a graph with $m$ edges. Then by induction

$$
\pi_k(G - e) = k^\nu + \sum_{i=1}^{\nu-1} (-1)^{\nu-i} a_i k^i
$$

$$
\pi_k(G \cdot e) = k^{\nu-1} + \sum_{i=1}^{\nu-2} (-1)^{\nu-1-i} b_i k^i
$$

where $a_1, \ldots, a_{\nu-1}, b_1, \ldots, b_{\nu-2}$ are non-negative integers. Then

$$
\pi_k(G) = k^\nu - (a_{\nu-1} + 1) k^{\nu-1} + \sum_{i=1}^{\nu-2} (-1)^{\nu-i} (a_i + b_i) k^i.
$$

\qed
Triangle free graphs with high chromatic number

**Theorem 6** For any positive integer $k$, there exists a triangle-free graph with chromatic number $k$.

**Proof** For $k = 1, 2$ we use $K_1, K_2$ respectively.

For larger $k$ we use induction on $k$. Suppose we have a triangle-free graph $G_k = (V_k, E_k)$ of chromatic number $k$. Let $V_k = \{v_1, v_2, \ldots, v_n\}$. Form $G_k$ as follows:

Add vertices $\{v\} \cup U = \{u_1, u_2, \ldots, u_n\}$ to $G_k$. Join $u_i$ to $v$ and the neighbours of $v_i$ in $G_k$, for $1 \leq i \leq n$. 
(a) $G_{k+1}$ has no triangles. 
$U$ is an independent set and so any triangle will have at most one vertex from $U$. Thus there are no triangles involving $v$. Finally, if $u_i, v_j, v_k$ is a triangle then $v_i, v_j, v_k$ is a triangle of $G_k$.

(b) $G_{k+1}$ does not have a proper $k$-colouring. 
Suppose there was one $c^*$. We can assume that $c^*(v) = k$ and then $U$ is coloured from $\{1, 2, \ldots, k-1\}$. But now we can define a proper $(k-1)$-colouring $c$ of $G_k$ by

$$c(v_i) = \begin{cases} 
    c^*(v_i) & \text{if } c^*(v_i) \neq k \\
    c^*(u_i) & \text{if } c^*(v_i) = k 
\end{cases}$$

This is a proper colouring of $G_k$ since if $v_i v_j$ is an edge of $G_k$ with $c(v_i) = c(v_j)$ then exactly one of $c(v_i) \neq c^*(v_i)$ or $c(v_j) \neq c^*(v_j)$ holds. Assume the former. Then $c^*(v_i) = k$ and $c(v_i) = c^*(u_i) \neq c^*(v_j) = c(v_j)$. Thus $G_{k+1}$ is $k$-colourable implies $G_k$ is $(k-1)$-colourable, which it isn’t.
(c) $G_{k+1}$ has a proper $(k + 1)$-colouring. Let $c$ be a proper $k$-colouring of $G_k$. Extend this to $U$ by putting $c(u_i) = c(v_i)$ and then let $c(v) = k + 1$. Note that $u_i$ and $v_i$ have the same colour and the same neighbours in $V_k$ and so the colouring remains proper.