Matchings

A matching $M$ of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.
An $M$-alternating path joining 2 $M$-unsaturated vertices is called an $M$-augmenting path.
$M$ is a maximum matching of $G$ if no matching $M'$ has more edges.

**Theorem 1**  
$M$ is a maximum matching iff $M$ admits no $M$-augmenting paths.

**Proof**  
Suppose $M$ has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where $e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and $f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.

$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$
• $|M'| = |M| + 1$.

• $M'$ is a matching

For $x \in V$ let $d_M(x)$ denote the degree of $x$ in matching $M$, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} 
    d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\
    d_M(x) & x \in \{b_1, \ldots, a_k\} \\
    d_M(x) + 1 & x \in \{a_0, b_{k+1}\}
\end{cases}$$

So if $M$ has an augmenting path it is not maximum.
Suppose $M$ is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \Delta M']$ where $M \Delta M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in exactly one of $M, M'$. 

Maximum degree of $H$ is 2, at most 1 edge from $M$ or $M'$. So $H$ is a collection of vertex disjoint alternating paths and cycles.

$|M'| > |M|$ implies that there is at least one path of type (d).

Such a path is $M$-augmenting
Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition $A, B$.

For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a, b) \in E\}$.

Clearly, $|M| \leq |A|, |B|$ for any matching $M$ of $G$. 
Hall’s Theorem

Theorem 2 \( G \) contains a matching of size \( |A| \) iff

\[
|N(S)| \geq |S| \quad \forall S \subseteq A. \tag{1}
\]

\( N(\{a_1, a_2, a_3\}) = \{b_1, b_2\} \) and so at most 2 of \( a_1, a_2, a_3 \) can be saturated by a matching.
**Only if:** Suppose $M = \{(a, \phi(a)) : \ a \in A\}$ saturates $A$.

\[
|N(S)| \geq |\{\phi(s) : s \in S\}| = |S| \tag{1}
\]

and so (1) holds.

**If:** Let $M = \{(a, \phi(a)) : \ a \in A'\} \ (A' \subseteq A)$ is a maximum matching. Suppose $a_0 \in A$ is $M$-unsaturated. We show that (1) fails.
Let

\[ A_1 = \{ a \in A : \text{such that } a \text{ is reachable from } a_0 \text{ by an } M\text{-alternating path.} \} \]

\[ B_1 = \{ b \in B : \text{such that } b \text{ is reachable from } a_0 \text{ by an } M\text{-alternating path.} \} \]
• $B_1$ is $M$-saturated else there exists an $M$-augmenting path.

• If $a \in A_1 \setminus \{a_0\}$ then $\phi(a) \in B_1$.

• If $b \in B_1$ then $\phi^{-1}(b) \in A_1 \setminus \{a_0\}$.

So

$$|B_1| = |A_1| - 1.$$ 

• $N(A_1) \subseteq B_1$

So

$$|N(A_1)| = |A_1| - 1$$

and (1) fails to hold.
Marriage Theorem

Theorem 3  Suppose $G = (A \cup B, E)$ is $k$-regular. ($k \geq 1$) i.e. $d_G(v) = k$ for all $v \in A \cup B$. Then $G$ has a perfect matching.

Proof

\[ k|A| = |E| = k|B| \]

and so $|A| = |B|$.

Suppose $S \subseteq A$. Let $m$ be the number of edges incident with $S$. Then

\[ k|S| = m \leq k|N(S)|. \]

So (1) holds and there is a matching of size $|A|$ i.e. a perfect matching.
**Edge Covers**

A set of vertices $X \subseteq V$ is a **covering** of $G = (V, E)$ if every edge of $E$ contains at least one endpoint in $X$.

![Diagram of a covering set](image)

**Lemma 1** *If $X$ is a covering and $M$ is a matching then $|X| \geq |M|$.*

**Proof** Let $M = \{(a_1, b_i) : 1 \leq i \leq k\}$. Then $|X| \geq |M|$ since $a_i \in X$ or $b_i \in X$ for $1 \leq i \leq k$ and $a_1, \ldots, b_k$ are distinct. \[\square\]
Konig’s Theorem

Let $\mu(G)$ be the maximum size of a matching. Let $\beta(G)$ be the minimum size of a covering. Then

$$\mu(G) \leq \beta(G).$$

Theorem 4 If $G$ is bipartite then $\mu(G) = \beta(G)$.

Proof Let $M$ be a maximum matching. Let $S_0$ be the $M$-unsaturated vertices of $A$. Let $S \supseteq S_0$ be the $A$-vertices which are reachable from $S$ by $M$-alternating paths. Let $T$ be the $M$-neighbours of $S \setminus S_0$. 
Let $X = (A \setminus S) \cup T$.

- $|X| = |M|$.
- $|T| = |S \setminus S_0|$. The remaining edges of $M$ cover $A \setminus S$ exactly once.

- $X$ is a cover.

There are no edges $(x, y)$ where $x \in S$ and $y \in B \setminus T$. Otherwise, since $y$ is $M$-saturated (no $M$-augmenting paths) the $M$-neighbour of $y$ would have to be in $S$, contradicting $y \notin T$. 

\[\square\]
**Tutte’s Theorem**

We now discuss arbitrary (i.e. non-bipartite) graphs. For $S \subseteq V$ we let $o(G - S)$ denote the number of components of odd cardinality in $G - S$.

**Theorem 5** $G$ has a perfect matching iff

$$o(G - S) \leq |S| \quad \text{for all } S \subseteq V.$$  \hfill (2)

**Proof** We restrict our attention to simple graphs.
Only if:

Suppose \(|S| = k\) and \(O_1, O_2, \ldots, O_{k+1}\) are odd components of \(G - S\). In any perfect matching of \(G\), at least one vertex \(x_i\) of \(C_i\) will have to be matched outside \(O_i\) for \(i = 1, 2, \ldots, k + 1\). But then \(x_1, x_2, \ldots, x_{k+1}\) will all have to be matched with \(S\), which is impossible.
**If:** Suppose (2) holds and $G$ has no perfect matching. Add edges until we have a graph $G^*$ which satisfies

- $G^*$ has no perfect matching.
- $G^* + e$ has a perfect matching for all $e \notin E(G^*)$.

Clearly,

$$o(G^* - S) \leq o(G - S) \leq |S| \quad \text{for all } S \subseteq V.$$  \hspace{1cm} (3)

In particular, if $S = \emptyset$, $o(G^*) = 0$ and $|V|$ is even.

$$U = \{v \in V : d_{G^*}(v) = v - 1\}.$$

$U \neq V$ else $G^*$ has a perfect matching.
**Claim:** $G^* - U$ is the disjoint union of complete graphs.

Suppose $C$ is a component of $G^* - U$ which is not a clique. Then there exist $x, y, z \in C$ such that $xy, xz \in E(G^*)$ and $xz \notin E(G^*)$.

Take $x, z \in C$ at distance 2 in $G^*$.

$y \notin U$ implies that there exists $w \notin U$ with $yw \notin E(G^*)$.

Let $M_1, M_2$ be perfect matchings in $G^* + xz, G^* + yw$ respectively.
Let $H = M_1 \Delta M_2$. $H$ is a collection of vertex disjoint even cycles.

**Case 1:** $xz, yw$ are in different cycles of $H$.

+ edges form a perfect matching in $G^*$ — contradiction.
**Case 2:** $xz, yw$ are in same cycle of $H$.

+ edges form a perfect matching in $G^* - contradiction.

Claim is proved.
Suppose $G - U$ has $\ell$ odd components. Then

- $\ell \leq |U|$ from (3).
- $\ell = |U| \mod 2$, since $|V|$ is even.

\[ G^* \text{ has a perfect matching} - \text{contradiction.} \]
**Petersen’s Theorem**

**Theorem 6** *Every 3-regular graph without cut-edges contains a perfect matching.*

**Proof** Suppose $S \subseteq V$. Let $G - S$ have components $C_1, C_2, \ldots, C_r$ where $C_1, C_2, \ldots, C_\ell$ are odd.

$m_i$ is the number of $C_i : S$ edges; $m_i \geq 2$. 

$n_i$ is the number of edges contained in $C_i$.

$$3|C_i| = m_i + 2n_i.$$ 

So $m_i$ is odd for $1 \leq i \leq \ell$. Hence $m_i \geq 3$ for $1 \leq i \leq \ell$. Thus

$$3\ell \leq m_1 + m_2 + \cdots + m_\ell \leq 3|S|,$$

and (2) holds. □