Eulerian Graphs

An Eulerian cycle of a graph $G = (V, E)$ is a closed walk which uses each edge $e \in E$ exactly once.

The walk using edges a,b,c,d,e,f,g,h,j,k in this order is an Eulerian cycle.
Theorem 1  A connected graph is Eulerian i.e. has an Eulerian cycle, iff it has no vertex of odd degree.

Proof  Suppose $W = (v_1, v_2, \ldots, v_m, v_1)$ $(m = |E|)$ is an Eulerian cycle. Fix $v \in V$. Whenever $W$ visits $v$ it enters through a new edge and leaves through a new edge. Thus each visit requires 2 new edges. Thus the degree of $v$ is even.
The converse is proved by induction on $|E|$. The result is true for $|E| = 3$. The only possible graph is a triangle.

Assume $|E| \geq 4$. $G$ is not a tree, since it has no vertex of degree 1. Therefore it contains a cycle $C$. Delete the edges of $C$. The remaining graph has components $K_1, K_2, \ldots, K_r$.

Each $K_i$ is connected and is of even degree – deleting $C$ removes 0 or 2 edges incident with a given $v \in V$. Also, each $K_i$ has strictly less than $|E|$ edges. So, by induction, each $K_i$ has an Eulerian cycle, $C_i$ say.

We create an Eulerian cycle of $G$ as follows: let $C = (v_1, v_2, \ldots, v_s, v_1)$. Let $v_{i_t}$ be the first vertex of $C$ which is in $K_t$. Assume w.l.o.g. that $i_1 < i_2 < \cdots < i_r$.

$$W = (v_1, v_2, \ldots, v_{i_1}, C_1, , v_{i_1}, \ldots, v_{i_2}, C_2, v_{i_2},$$
$$\ldots, v_{i_r}, C_r, v_{i_r}, \ldots, v_1)$$

is an Eulerian cycle of $G$. □
Corollary 1  A connected graph has an Eulerian Walk i.e. a walk which uses each edge exactly once, iff it has exactly 2 vertices of odd degree.

Proof  If a walk exists then the endpoints have odd degree and the interior vertices have even degree. Conversely, if there are two odd degree vertices \( x, y \) add an extra edge \( e = xy \) to create a connected graph \( G' \) with only even vertices. This has an Eulerian cycle \( C \). Delete \( e \) from \( C \) to create the required path. \( \square \)
Hamilton Cycles

A Hamilton Cycle of a graph $G = (V, E)$ is a cycle which goes through each vertex (once).

A graph is called Hamiltonian if it contains a Hamilton cycle.

Hamiltonian Graph  Non-Hamiltonian Graph

Petersen Graph
Lemma 1 Let $G = (V, E)$ and $|V| = n$. Suppose $x, y \in V$, $e = (x, y) \notin E$ and $d(x) + d(y) \geq n$. Then

$G + e$ is Hamiltonian $\iff G$ is Hamiltonian.

Proof

$\Leftarrow$ Trivial.

$\implies$ Suppose $G + e$ has a Hamilton cycle $H$. If $e \notin H$ then $H \subseteq G$ and $G$ is Hamiltonian.

Suppose $e \in H$. We show that we can find another Hamilton cycle in $G + e$ which does not use $e$. 
Thus $|S \cap T| = |S| + |T| - |S \cup T| \geq 1$ and so $\exists 1 \neq k \in S \cap T$ and then

$$H' = (v_1, v_2, \ldots, v_k, v_n, v_{n-1}, \ldots, v_{k+1}, v_1)$$

is a Hamilton cycle of $G$. 

$$H = (x = v_1, v_2, \ldots, v_n = y, x).$$

$S = \{i: (x, v_{i+1}) \in E\}$ and 

$T = \{i: (y, v_i) \in E\}$.

$S \subseteq \{1, 2, \ldots, n - 2\}$, $T \subseteq \{2, 3, \ldots, n - 1\}$. 

$|S| + |T| \geq n$ and $|S \cup T| \leq n - 1$. 

$|S \cap T| = |S| + |T| - |S \cup T| \geq 1$
Bondy-Chvátal Closure of a graph

begin
\[ c(G) := G \]
while \( \exists (x, y) \notin E \) with \( d_{c(G)}(x) + d_{c(G)}(y) \geq n \) do
begin
\[ c(G) := c(G) + (x, y) \]
end
Output \( c(G) \)
end

The graph \( c(G) \) is called the closure of \( G \).
Lemma 2 \( c(G) \) is independent of the order in which edges are added i.e. it depends only on \( G \).

Proof Suppose algorithm is run twice to obtain 
\[ G_1 = G + e_1 + e_2 + \cdots + e_k \]  
and 
\[ G_2 = G + f_1 + f_2 + \cdots + f_\ell. \]
We show that \( \{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_\ell\} \).

Suppose not. Let \( t = \min\{i : e_i \notin G_2\} \), \( e_t = (x, y) \) and \( G' = G + e_1 + e_2 + \cdots + e_{t-1} \). Then 
\[ d_{G_2}(x) + d_{G_2}(y) \geq d_{G'}(x) + d_{G'}(y) \geq n \]
since \( e_t \) was added to \( G' \).

But then \( e_t \) should have been added to \( G_2 \) — contradiction.
\begin{itemize}
  \item $c(G')$ Hamiltonian $\Rightarrow G$ is Hamiltonian.
  \item $c(G')$ complete $\Rightarrow G$ is Hamiltonian.
  \item $\delta(G) \geq n/2$ $\Rightarrow G$ is Hamiltonian.
\end{itemize}

Second statement is due to Bondy and Murty. Third statement is due to Dirac.
Theorem 2  Let $G$ be a simple graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_\nu$, $\nu \geq 3$. Suppose that there does not exist $m < \nu/2$ such that

$$d_m \leq m \text{ and } d_{\nu-m} < \nu - m.$$

Then $G$ is Hamiltonian.

Proof  We prove that $c(G)$ is complete. Let $d'$ denote degree in $c(G)$. Suppose $c(G)$ is not complete. Among all pairs of vertices $u, v$ which are not adjacent in $c(G)$ choose a pair which maximise $d'(u) + d'(v)$ and assume $m = d'(u) \leq d'(v)$. Note that

$$d'(u) + d'(v) \leq \nu - 1.$$

$S = \{w \in V \setminus \{v\} : v, w \text{ not adjacent in } c(G)\}$.  
$T = \{w \in V \setminus \{u\} : u, w \text{ not adjacent in } c(G)\}$.  

$$|S| = \nu - 1 - d'(v) \geq d'(u) = m \quad (1)$$

$$|T \cup \{u\}| = \nu - m. \quad (2)$$
The choice of $u, v$ means that

\begin{align*}
  d'(w) &\leq d'(u) \quad \text{for } w \in S \quad (3) \\
  d'(w) &\leq d'(v) < \nu - m \quad \text{for } w \in T \quad (4)
\end{align*}

Now $d(w) \leq d'(w)$ for $w \in V$ and so

(1) and (3) imply that $d_m \leq m$.

(2) and (4) imply that $d_{\nu-m} < \nu - m$.

Contradiction. \qed