Connectivity

$G$ is $k$-connected if $S \subseteq V, |S| < k$ implies $G - S$ is connected.

$$\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}.$$
$G$ is $k$-edge connected if $S \subseteq E, |S| < k$ implies $G - S$ is connected.

$$\kappa'(G) = \max \{k : G \text{ is } k\text{-edge connected}\}.$$
Assume $G$ connected.

$S$ is a $k$-vertex cutset if $S \subseteq V, |S| = k$ and $G - S$ is not connected.

A 1-vertex cutset is a cutpoint.

$S$ is a $k$-edge cutset if $S \subseteq E, |S| = k$ and $G - E$ is not connected.

A 1-edge cutset is a bridge or cut-edge.
Lemma 1 \textit{If }$G$\textit{ is connected and }$e$\textit{ is a bridge, then }$H = G - e$\textit{ has exactly 2 components.}

\textbf{Proof} \hspace{0.5cm} \textit{If }$H$\textit{ has components }$C_1, C_2, C_3$\textit{ then }$G = H + e$\textit{ has }$\geq 2$\textit{ components, since adding an edge decreases the number of components by at most 1. This contradicts the fact that }$G$\textit{ is connected.} $\Box$
Complete Graphs

$K_n$ has no vertex cutsets.

$\kappa(K_n) = n - 1$ by convention.

$\kappa'(K_n) = n - 1$.

So in general

$\kappa(G) \leq \nu - 1$.

$G$ not complete. $v, w$ not neighbours.
$V \setminus \{v, w\}$ is a $(\nu - 2)$-vertex cutset.
Theorem 1

\[ \kappa(G) \leq \kappa'(G) \leq \delta(G). \]

Proof  If \( G \) has no edges then

\[ \kappa' = 0 = \delta. \]

Otherwise the set of edges incident with a vertex \( v \) of minimum degree is a \( \delta \)-edge cutset.

Therefore \( \kappa' \leq \delta \).
We prove that $\kappa \leq \kappa'$ by induction on $\kappa'$.

True for $\kappa' = 0$.

Assume true for all graphs with $\kappa' < k$ and let $G$ be a graph with $\kappa'(G) = k$.

Suppose $A \subseteq E$ is a $k$-edge cutset of $G$.

Let $e \in A$ and $H = G - e$. Then

$$H - (A \setminus e) = G - A$$

is not connected

and so $\kappa'(H) < k$.

By the induction hypothesis $\kappa(H) \leq \kappa'(H) \leq k - 1$.

Let $S \subseteq V$ be a $k - 1$-vertex cutset of $H$. 
If $G - S$ is not connected then $\kappa(G) \leq k - 1 < \kappa'(G)$.

Assume therefore that $G - S$ is connected.

Neither endpoint of $e$ is in $S$, else $G - S = H - S$.

$e$ is a bridge of $G - S$ since $(G - S) - e = H - S$ is not connected. It has 2 components $C_1, C_2$. 
If $|C_1| \geq 2$ then $S + v$ is a $k$-vertex cutset of $G$ and so $\kappa(G) \leq k$.

Similarly if $|C_2| \geq 2$.

So assume that $G - S$ is the just the edge $vw$.

Then $\nu(G) = k + 1$ and so $\kappa(G) \leq k$. 
A block is a connected graph with no cutpoints.

Thus a block is either a single vertex, a single edge or if $\nu \geq 3$ it is a 2-connected graph.

A block of a graph is a maximal connected subgraph with no cutpoints.

Note that blocks partition the edges of $G$, not the vertices.
Union and Intersection of Graphs

\( G_i = (V_i, E_i), i = 1, 2. \)

\[ G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2). \]

\[ G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2) \]

provided \( V_1 \cap V_2 \neq \emptyset. \)

**Theorem 2** \( A \) connected graph \( G \) can be expressed as

\[ G = B_1 \cup B_2 \cup \cdots B_r \]

where \( B_1, B_2, \ldots, B_r \) are the blocks of \( G \).

By induction on \( \nu \). Trivial for \( \nu = 1. \)
Assume true for all connected graphs with \( \nu < k \) and suppose that \( G \) has \( k \) vertices.
(a) $G$ has no cutpoints – $G = B_1$.

(b) $G$ has a cutpoint $v$.

Let $G - v$ have components $C_1, C_2, \ldots C_s$. 
$C_i + v$ is connected for $1 \leq i \leq s$.

By induction

$$C_i + v = \bigcup_{j=1}^{k_i} B_{i,j}$$

where the $B_{i,j}$ are the blocks of $C_i + v$.

Thus

$$G = \bigcup_{i=1}^{r} \bigcup_{j=1}^{k_i} B_{i,j}.$$ 

We still have to check that the $B_{i,j}$ are maximal 2-connected subgraphs. But $B_{i,j}$ is not strictly contained in any 2-connected subgraph of $C_i + v$ since it is a block of $C_i + v$. Also, if $x \notin C_i + v$ then every path from $x$ to $C_i$ must go through $v$ and so $v$ is a cutpoint of any subgraph containing $B_{i,j}$ and $x$. 
Theorem 3  If $B_1, B_2$ are blocks of the connected graph $G$ then

$$|V(B_1) \cap V(B_2)| \leq 1.$$  

Proof  Suppose that $|V(B_1) \cap V(B_2)| \geq 2$. We obtain the contradiction that $B_1 \cup B_2$ is 2-connected. Let $x \in V(B_1) \cup V(B_2)$ and $y \in (V(B_1) \cap V(B_2)) - x$. Then there is a path in $B_i$ from every vertex $v$ of $B_i - x$ to $y$. Thus $B_1 \cup B_2 - x$ is connected. \[\square\]
Lemma 2  If $v$ is a cutpoint of connected graph then there exist blocks $B_1, B_2$ such that $V(B_1) \cap V(B_2) = \{v\}$.

Proof  Let $G-v$ have components $C_1, C_2, \ldots , C_r$. Let $B_i$ be the block of $C_i + v$ which contains $v$ for $i = 1, 2$. $B_1, B_2$ are blocks of $G$, by the same argument as given in end of proof of Theorem 2. $\square$
Lemma 3  If $B_1 \neq B_2$ are blocks of $G$ and $V(B_1) \cap V(B_2) = \{v\}$ then every path from a vertex of $B_1$ to a vertex of $B_2$ goes through $v$.

Proof  If there is a path $P$ from $x \in B_1$ to $y \in B_2$ which avoids $v$ then $B_1 \cup B_2 \cup P$ is 2-connected.

Corollary 1  If $B_1 \neq B_2$ are blocks of $G$ and $V(B_1) \cap V(B_2) = \{v\}$ then $v$ is a cutpoint of $G$. 
Lemma 4  Suppose $B_1 \neq B_2$ are vertex disjoint blocks of $G$ and there is an edge $xy$ with $x \in V(B_1)$, $y \in V(B_2)$. Then $x, y$ forms a block of $G$ and both of $x, y$ are cutpoints.

Proof   If $y$ is of degree 1 then $B_1 = x$ is not a block.

\[\text{\begin{tikzpicture}
    
    
    
    \end{tikzpicture}}\]

Otherwise if $y$ is not of degree 1 and is not a cutpoint then there is a path $P$ from $B_2$ to $B_1 - x$ which implies $B_1 \cup B_2 \cup P$ is 2-connected – contradiction.

\[\text{\begin{tikzpicture}
    
    \end{tikzpicture}}\]

Thus $xy$ is a bridge of $G$ and so is a block.
Let $G$ be a connected graph with blocks $B_1, B_2, \ldots , B_r$ and cutpoints $c_1, c_2, \ldots , c_s$. We define a bipartite graph $H$ with $V(H) = \{b_1, b_2, \ldots , b_r\} \cup \{c_1, c_2, \ldots , c_s\}$ (block vertices and cut vertices) and $E(H) = \{b_ic_j : c_j \in B_i\}$.

**Theorem 4** $H$ is a tree.
Proof  If $G$ is 2-connected then $H$ consists of a single vertex. Assume that $G$ is not 2-connected.

(a) $H$ is connected.
Suppose that $H$ contains components $C_1, C_2, \ldots, C_r$, $r \geq 2$. Each component contains at least one block vertex and at least one cut vertex – each block contains a cutpoint and each cutpoint is contained in a block.

Since $G$ is connected, there exist $C_i, C_j$ and $b_i \in C_i, b_j \in C_j$ such that either

1. $V(B_i) \cap V(B_j) = \{c\}$: but then $H$ contains the path $b_i, c, b_j$ – contradiction.

2. $V(B_i) \cap V(B_j) = \emptyset$ and there is an edge $x, y$ joining $B_i$ to $B_j$: but then $x, y$ is a block $B_k$, say, and $H$ contains the path $b_i, x, b_k, y, b_j$ – contradiction.
(b) $H$ contains no cycles.

If $H$ contains the cycle $(b_1, c_1, b_2, c_2, \ldots, b_k, c_k, b_1)$ then $B_1 \cup B_2 \cup \cdots B_k$ is 2-connected – contradiction.
A family of paths is said to be *internally disjoint* if no vertex of $G$ is an internal vertex of more than one path.

Theorem 5 A graph $G$ with $\nu \geq 3$ is 2-connected iff any two vertices of $G$ are connected by at least two internally disjoint paths.
Proof

If: $G - v$ is connected for any $v$ since $x, y \in V - v$ must contain at least one path which avoids $v$.

Only if: assume $G$ is 2-connected. We show by induction on $d(u, v)$ that every pair of vertices $u, v$ are joined by two internally disjoint paths.

Base case: $d(u, v) = 1$.

$e = uv$ is not a cut-edge and so is contained in a cycle.
Assume true for $d(u, v) < k$ and consider a pair $u, v$ with $d(u, v) = k$. Let $w$ be the penultimate vertex on some path of length $k$ from $u$ to $v$. Thus $d(u, w) = k - 1$ and there are two internally disjoint paths $P, Q$ joining $u$ and $w$.

$G - w$ is connected and so there exists a $u, v$-path $P'$ in $G - w$. Let $x$ be the last vertex of $P'$ which is not in $P \cup Q$.

If $x \in P$ take $P(u, x) + P'(x, v)$ and $Q$ as the two internally disjoint paths.
Corollary 2  If $G$ is 2-connected and $\nu \geq 3$ then every pair of vertices are contained in a cycle.

Corollary 3  If $G$ is 2-connected and $\nu \geq 3$ then every pair of edges $e_1, e_2$ are contained in a cycle.

$G'$ is obtained from $G$ by dividing $e_1, e_2$. $G'$ is 2-connected. Apply Theorem 5.