Directed graphs

Digraph $D = (V, A)$.  
$V=$\{vertices\},  $A=$\{arcs\}

V={a,b,...,h},  A={(a,b),(b,a),...}  
(2 arcs with endpoints (c,d))

Thus a digraph is a graph with oriented edges.  
$D$ is *strict* if there are no loops or repeated edges.
Digraph $D$: $G(D)$ is the underlying graph obtained by replaced each arc $(a, b)$ by an edge $\{a, b\}$.

The graph underlying the digraph on previous slide
Graph $G$: an orientation of $G$ is obtained by replacing each edge $\{a, b\}$ by $(a, b)$ or $(b, a)$.

There are $2^{|E|}$ distinct orientations of $G$. 

$G$ 

Orientation of $G$
Walks, trails, paths, cycles now have directed counterparts.

Directed Walk: (c,d,e,f,a,b,g,f).
Directed Path: (a,b,g,f).
Directed Cycle: (g,a,b,a)

(e,f,g,a) is not a directed walk -- there is no arc (f,g).
The *indegree* $d_D^-(v)$ of vertex $v$ is the number of arcs $(x,v)$, $x \in V$. The *outdegree* $d_D^+(v)$ of vertex $v$ is the number of arcs $(v,x)$, $x \in V$.

![Graph with vertices and arrows]

Note that since each arc contributes one to a vertex outdegree and one to a vertex indegree,

$$\sum_{v \in V} d_D^+(v) = \sum_{v \in V} d_D^-(v) = |A|.$$
Strong Connectivity or Diconnectivity

Given digraph $D$ we define the relation $\sim$ on $V$ by $v \sim w$ iff there is a directed walk from $v$ to $w$ and a directed walk from $w$ to $v$.

This is an equivalence relation (proof same as directed case) and the equivalence classes are called strong components or dicomponents.

Here the strong components are

$$\{a, b, g\}, \{c\}, \{d\}, \{e, f, h\}.$$
A graph is *strongly connected* if it has one strong component i.e. if there is a directed walk between each pair of vertices.

For a set \( S \subseteq V \) let

\[
N^+(S) = \{ w \notin S : \exists v \in S \text{ s.t.} (v, w) \in A \}.
\]

\[
N^-(S) = \{ w \notin S : \exists v \in S \text{ s.t.} (w, v) \in A \}.
\]

**Theorem 1**  
*D is strongly connected iff there does not exist* \( S \subseteq V, S \neq \emptyset, V \) *such that* \( N^+(S) = \emptyset \).

**Proof**  
*Only if:* suppose there is such an \( S \) and \( x \in S, y \in V \setminus S \) and suppose there is a directed walk \( W \) from \( x \) to \( y \). Let \( (v_1 = x, v_2, \ldots, v_k = y) \) be the sequence of vertices traversed by \( W \). Let \( v_i \) be the first vertex of this sequence which is not in \( S \). Then \( v_i \in N^+(S) \), contradiction, since arc \( (v_{i-1}, v_i) \) exists.
**If:** suppose that $D$ is not strongly connected and that there is no directed walk from $x$ to $y$. Let $S = \{v \in V : \exists$ a directed walk from $x$ to $v\}$.

$S \neq \emptyset$ as $x \in S$ and $S \neq V$ as $y \notin S$.

Then $N^+(S) = \emptyset$. If $z \in N^+(S)$ then there exists $w \in S$ such that $(w, z) \in A$. But then since $w \in S$ there is a directed walk from $x$ to $w$ which can be extended to $z$, contradicting the fact that $z \notin S$. $\square$
A Directed Acyclic Graph (DAG) is a digraph without any directed cycles.

**Lemma 1** If $D$ is a DAG then $D$ has at least one source (vertex of indegree 0) and at least one sink (vertex of outdegree 0).

**Proof** Let $P = (v_1, v_2, \ldots, v_k)$ be a directed path of maximum length in $D$. Then $v_1$ is a source and $v_k$ is a sink.
Suppose for example that there is an edge $xv_1$. Then either

(a) $x \not\in \{v_2, v_3, \ldots, v_k\}$. But then $(x, P)$ is a longer directed path than $P$ – contradiction.

(b) $x = v_i$ for some $i \neq 1$ and $D$ contains the cycle $v_1, v_2, \ldots, v_i, v_1$.  

\[ \square \]

A topological ordering $v_1, v_2, \ldots, v_\nu$ of the vertex set of a digraph $D$ is one in which

$$v_i v_j \in A \text{ implies } i < j.$$
**Theorem 2**  
\( D \) has a topological ordering iff \( D \) is a DAG.

**Proof**  
**Only if:** Suppose there is a topological ordering and a directed cycle \( v_{i_1}, v_{i_2}, \ldots, v_{i_k} \). Then

\[
i_1 < i_2 < \cdots < i_k < i_1
\]

which is absurd.

**if:** By induction on \( \nu \). Suppose that \( D \) is a DAG. The result is true for \( \nu = 1 \) since \( D \) has no loops. Suppose that \( \nu > 1 \), \( v_{\nu} \) is any sink of \( D \) and let \( D' = D - v_{\nu} \).

\( D' \) is a DAG and has a topological ordering \( v_1, v_2, \ldots, v_{\nu-1} \), induction. \( v_1, v_2, \ldots, v_{\nu} \) is a topological ordering of \( D \). For if there is an edge \( v_i v_j \) with \( i > j \) then (i) it cannot be in \( D' \) and (ii) \( i \neq \nu \) since \( v_{\nu} \) is a sink.

\( \Box \)
Theorem 3 Let $G = G(D)$. Then $D$ contains a directed path of length $\chi(G) - 1$.

Proof Let $D = (V, A)$ and $A' \subseteq A$ be a minimal set of edges such that $D' = D - A$ is a DAG.

Let $k$ be the length of the longest directed path in $D'$.

Define $c(v) =$ length of longest path from $v$ in $D'$.
$c(v) \in \{0, 1, 2, \ldots, k\}$. We claim that $c(v)$ is a proper colouring of $G$, proving the theorem.
Note first that if $D'$ contains a path $P = (x_1, x_2, \ldots, x_k)$ then
\[ c(x_1) \geq c(x_k) + k - 1. \quad (1) \]
(We can add the longest path $Q$ from $x_k$ to $P$ to create a path $(P, Q)$. This uses the fact that $D'$ is a DAG.)

Suppose $c$ is not a proper colouring of $G$ and there exists an edge $vw \in G$ with $c(v) = c(w)$. Suppose $vw \in A$ i.e. it is directed from $v$ to $w$.

**Case 1:** $vw \notin A'$. $(1)$ implies $c(v) \geq c(w) + 1$ – contradiction.

**Case 2:** $vw \in A'$. There is a cycle in $D' + vw$ which contains $vw$, by the minimality of $A'$. Suppose that $C$ has $\ell \geq 2$ edges. Then $(1)$ implies that $c(w) \geq c(v) + \ell - 1$. \qed
Tournaments

A tournament is an orientation of a complete graph $K_n$.

1, 2, 5, 4, 3 is a directed Hamilton Path

Corollary 1  A tournament $T$ contains a directed Hamilton path.

Proof  $\chi(G(T)) = n$. Now apply Theorem 3.  \qed
**Theorem 4** If $D$ is a strongly connected tournament with $\nu \geq 3$ then $D$ contains a directed cycle of size $k$ for all $3 \leq k \leq \nu$.

**Proof** By induction on $k$.

$k = 3$.
Choose $v \in V$ and let $S = N^+(V)$, $T = N^-(v) = V \setminus (S \cup \{v\})$.

![Diagram](image)

$S \neq \emptyset$ since $D$ is strongly connected. Similarly, $S \neq V \setminus \{v\}$ else $N^+(V \setminus \{v\}) = \emptyset$.

Thus $N^+(S) \neq \emptyset$. $v \notin N^+(S)$ and so $N^+(S) = T$. Thus $\exists x \in S, y \in T$ with $xy \in A$. 

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Suppose now that there exists a directed cycle $C = (v_1, v_2, \ldots, v_k, v_1)$.

**Case 1:** $\exists w \not\in C$ and $i \neq j$ such that $v_iw \in A$, $wv_j \in A$. 

![Diagram of directed cycle](image)
It follows that there exists $\ell$ with $v_\ell w \in A$, $wv_{\ell+1} \in A$.

$C' = (w, v_{\ell+1}, \ldots, v_\ell, v_1, \ldots, v_\ell, w)$ is a cycle of length $k + 1$. 
Case 2 \( V \setminus C = S \cup T \) where
\[
\begin{align*}
w \in S & \implies wv_i \in A, \ 1 \leq i \leq k. \\
w \in T & \implies v_iw \in A, \ 1 \leq i \leq k.
\end{align*}
\]
\( S = \emptyset \) implies \( T = \emptyset \) (and \( C \) is a Hamilton cycle) or \( N^+(T) = \emptyset \).

\( T = \emptyset \) implies \( N^+(C) = \emptyset \).

Thus we can assume
\( S, T \neq \emptyset \) and \( N^+(T) \neq \emptyset \).
\( N^+(T) \cap C = \emptyset \) and so \( N^+(T) \cap S \neq \emptyset \).

Thus \( \exists x \in T, y \in S \) such that \( xy \in A \).

The cycle \( (v_1, x, y, v_3, \ldots, v_k, v_1) \) is a cycle of length \( k + 1 \).
Robbin’s Theorem

Theorem 5  A connected graph $G$ has an orientation which is strongly connected iff $G$ is 2-edge connected.

Only if: Suppose that $G$ has a cut edge $e = xy$.

If we orient $e$ from $x$ to $y$ (resp. $y$ to $x$) then there is no directed path from $y$ to $x$ (resp. $x$ to $y$).
**If:** Suppose $G$ is 2-edge connected. It contains a cycle $C$ which we can orient to produce a directed cycle.

At a general stage of the process we have a set of vertices $S \supseteq C$ and an orientation of the edges of $G[S]$ which is strongly connected.

If $S \neq V$ choose $x \in S$, $y \notin S$.

There are 2 edge disjoint paths $P_1, P_2$ joining $y$ to $x$.

Let $a_i$ be the first vertex of $P_i$ which is in $S$.

Orient $P_1[y, a_1]$ from $y$ to $a_1$.

Orient $P_2[y, a_2]$ from $a_2$ to $y$.
**Claim:** The subgraph $G[S \cup P_1 \cup P_2]$ is strongly connected.

Let $S' = S \cup P_1 \cup P_2$. We must show that there is a directed path from $\alpha$ to $\beta$ for all $\alpha, \beta \in S'$.

(i) $\alpha, \beta \in S$: $\exists$ a directed path from $\alpha$ to $\beta$ in $S$.

(ii) $\alpha \in S$, $\beta \in P_1 \setminus S$: Go from $\alpha$ to $a_2$ in $S$, from $a_2$ to $y$ on $P_2$, from $y$ to $\beta$ along $P_1$.

(iii) $\alpha \in S$, $\beta \in P_2 \setminus S$: Go from $\alpha$ to $a_2$ in $S$, from $a_2$ to $\beta$ on $P_2$.

(iv) $\alpha \in P_1 \setminus S$, $\beta \in S$: Go from $\alpha$ to $\alpha_1$ on $P_1$, from $a_1$ to $\beta$ in $S$.

(v) $\alpha \in P_2 \setminus S$, $\beta \in S$: Go from $\alpha$ to $y$ on $P_1$, from $y$ to $a_1$ on $P_1$, from $a_1$ to $\beta$ in $S$.

Continuing in this way we can orient the whole graph. 

\[\Box\]
Directed Euler Tours

An Euler tour of a digraph $D$ is a directed walk which traverses each arc of $D$ exactly once.

**Theorem 6** A digraph $D$ has an Euler tour iff $G(D)$ is connected and $d^+(v) = d^-(v)$ for all $v \in V$.

**Proof** This is similar to the undirected case.

*If:* Suppose $W = (v_1, v_2, \ldots, v_m, v_1)$ ($m = |A|$) is an Euler Tour. Fix $v \in V$. Whenever $W$ visits $v$ it enters through a new arc and leaves through a new arc. Thus each visit requires one entering arc and one leaving arc. Thus $d^+(v) = d^-(v)$.

*Only if:* We use induction on the number of arcs. $D$ is not a DAG as it has no sources or sinks. Thus it must have a directed cycle $C$. Now remove the edges of $C$. Each component $C_i$ of $G(D - C)$ satisfies the degree conditions and so contains an Euler tour $W_i$. Now, as in the undirected case, go round the cycle $C$ and the first time you visit $C_i$ add the tour $W_i$. This produces an Euler tour of the whole digraph $D$.  

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As a simple application of the previous theorem we consider the following problem. A 0-1 sequence \( x = (x_1, x_2, \ldots, x_m) \) has property \( P_n \) if for every 0-1 sequence \( y = (y_1, y_2, \ldots, y_n) \) there is an index \( k \) such that \( x_k = y_1, x_{k+1} = y_2, \ldots, x_{k+n-1} = y_n \). Here \( x_t = x_{m+1-t} \) if \( t > m \).

Note that we must have \( m \geq 2^n \) in order to have a distinct \( k \) for each possible \( x \).

**Theorem 7** There exists a sequence of length \( 2^n \) with property \( P_n \).

**Proof** Define the digraph \( D_n \) with vertex set \( \{0, 1\}^{n-1} \) and \( 2^n \) directed arcs of the form 
\( \left((p_1, p_2, \ldots, p_{n-1}), (p_2, p_3, \ldots, p_n)\right) \).

\( G(D_n) \) is connected as we can join \((p_1, p_2, \ldots, p_{n-1})\) to \((q_1, q_2, \ldots, q_{n-1})\) by the path \((p_1, p_2, \ldots, p_{n-1})\), \((p_2, p_3 \ldots, p_{n-1}, q_1)\), \((p_3, p_4, \ldots, p_{n-1}, q_1, q_2)\), \ldots, \((q_1, q_2, \ldots, q_{n-1})\). Each vertex of \( D_n \) has indegree and outdegree 2 and so it has an Euler tour \( W \).
Suppose that $W$ visits the vertices of $D_n$ in the sequence $(v_1, v_2, \ldots, v_{2^n})$. Let $x_i$ be the first bit of $v_i$. We claim that $x_1, x_2, \ldots, x_{2^n}$ has property $P_n$. Give arc $((p_1, p_2, \ldots, p_{n-1}), (p_2, p_3, \ldots, p_n))$ the label $(p_1, p_2, \ldots, p_n)$. No other arc has this label.

Given $(y_1, y_2, \ldots, y_n)$ let $k$ be such that $(v_k, v_{k+1})$ has this label. Then $v_k = (y_1, y_2, \ldots, y_{n-1})$ and $v_{k+1} = (y_2, y_3, \ldots, y_n)$ and then $x_k = y_1, x_{k+1} = y_2, \ldots, x_{k+n-1} = y_n$.  \qed