Department of Mathematics
Carnegie Mellon University

21-228 Discrete Mathematics
Test 3

Name:______________________________________

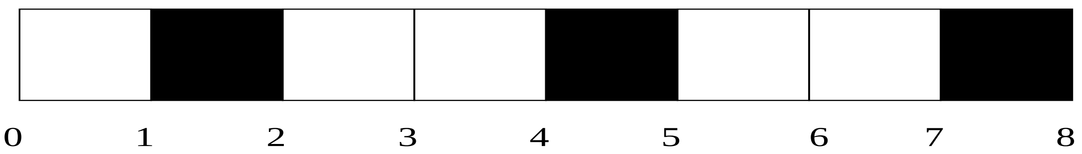
Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
Use generating functions to solve the recurrence

$$u_n - 6u_{n-1} + 9u_{n-2} = 3^n \qquad\qquad n \geq 2,$$

$u_0 = 1, u_1 = 9.$

Q2: (33pts)
You are given a set of unit length indistinguishable dominos. A domino can be placed at a point $i = 0, 1, \ldots, n$ of the real line to cover the interval $[i, i+1]$. Let $m_{n,k}$ be the number of ways of placing k dominos on the line so that no two dominos touch.
The diagram below shows a possible placement of 3 dominos when $n = 8$.



Show that
(a) $m_{n,0} = 1$.
(b) $m_{n,1} = n$.
(c) $m_{n,2} = \binom{n-1}{2}$.
(d) $m_{n,k} = m_{n-1,k} + m_{n-2,k-1}$ for $n \geq 2, k \geq 1$.

Q3: (34pts)
There are m **distinguishable** balls labelled $1, 2, \ldots, m$ and n colours.
(a) Suppose that $S \subseteq \{1, 2, \ldots, m-1\}$. Show that the number of ways of colouring the balls such that for $i \in S$, the colour of ball i is the same as the colour of ball $i+1$ is precisely $n^{m-|S|}$.
(b) Using (a) and the principle of inclusion-exclusion show that the number of ways of colouring the balls so that the colour of ball i is not the same as the colour of ball $i+1$ for $i = 1, 2, \ldots, m-1$ is

$$\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} n^{m-k}.$$