NETWORK FLOWS
A network consists of a loopless digraph \( D = (V, A) \) plus a function \( c : A \to R_+ \). Here \( c(x, y) \) for \( (x, y) \in A \) is the capacity of the edge \((x, y)\).

We use the following notation: if \( \phi : A \to R \) and \( S, T \) are (not necessarily disjoint) subsets of \( V \) then

\[
\phi(S, T) = \sum_{x \in S, y \in T} \phi(x, y).
\]

Let \( s, t \) be distinct vertices. An \( s - t \) flow is a function \( f : A \to R \) such that

\[
f(v, V \setminus \{v\}) = f(V \setminus \{v\}, v) \quad \text{for all } v \neq s, t.
\]

In words: flow into \( v \) equals flow out of \( v \).
An $s - t$ flow is *feasible* if

$$0 \leq f(x, y) \leq c(x, y) \quad \text{for all } (x, y) \in A.$$ 

An $s - t$ cut is a partition of $V$ into two sets $S, \bar{S}$ such that $s \in S$ and $t \in \bar{S}$.

The *value* $v_f$ of the flow $f$ is given by

$$v_f = f(s, V \setminus \{s\}) - f(V \setminus \{s\}, s).$$

Thus $v_f$ is the net flow leaving $s$.

The *capacity* of the cut $S : \bar{S}$ is equal to $c(S, \bar{S})$. 
Theorem

\[ \max \nu_f = \min \ c(S, \bar{S}) \]

where the maximum is over feasible \( s \rightarrow t \) flows and the minimum is over \( s \rightarrow t \) cuts.

Proof

We observe first that

\[
f(S, \bar{S}) - f(\bar{S}, S) = (f(S, N) - f(S, S)) - (f(N, S) - f(S, S))
\]

\[= f(S, N) - f(N, S)\]

\[= \nu_f + \sum_{v \in S \setminus \{s\}} (f(v, N) - f(N, v))\]

\[= \nu_f.\]

So,

\[\nu_f \leq f(S, \bar{S}) \leq c(S, \bar{S}).\]
This implies that
\[ \max v_f \leq \min c(S, \bar{S}). \quad (1) \]

Given a flow \( f \) we define a flow augmenting path \( P \) to be a sequence of distinct vertices \( x_0 = s, x_1, x_2, \ldots, x_k = t \) such that for all \( i \), either

\begin{align*}
F_1 & \quad (x_i, x_{i+1}) \in A \text{ and } f(x_i, x_{i+1}) < c(x_i, x_{i+1}), \\
F_2 & \quad (x_{i+1}, x_i) \in A \text{ and } f(x_{i+1}, x_i) > 0.
\end{align*}

If \( P \) is such a sequence, then we define \( \theta_P > 0 \) to be the minimum over \( i \) of \( c(x_i, x_{i+1}) - f(x_i, x_{i+1}) \) (Case (F1)) and \( f(x_{i+1}, x_i) \) (Case (F2)).
Claim 1: $f$ is a maximum value flow, iff there are no flow augmenting paths.

Proof: If $P$ is flow augmenting then define a new flow $f'$ as follows:

1. $f'(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \theta_P$ or
2. $f'(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \theta_P$
3. For all other edges, $(x, y)$, we have $f'(x, y) = f(x, y)$.

We can see that the flow stays balanced at $x_i$. 
We can see then that if there is a flow augmenting path then the new flow satisfies

\[ v_{f'} = v_f + \theta_P > v_f. \]

Let \( S_f \) denote the set of vertices \( v \) for which there is a sequence \( x_0 = s, x_1, x_2, \ldots, x_k = v \) which satisfies F1, F2 of the definition of flow augmenting paths.

If \( t \in S_f \) then the associated sequence defines a flow augmenting path. So, assume that \( t \notin S_f \). Then we have,

1. \( s \in S_f \).
2. If \( x \in S_f, y \in \bar{S}_f, (x, y) \in A \) then \( f(x, y) = c(x, y) \), else we would have \( y \in S_f \).
3. If \( x \in S_f, y \in \bar{S}_f, (y, x) \in A \) then \( f(y, x) = 0 \), else we would have \( y \in S_f \).
We therefore have

\[ v_f = f(S_f, \bar{S}_f) - f(\bar{S}_f, S) = c(S, \bar{S}_f). \]

We see from this and (1) that \( f \) is a flow of maximum value and that the cut \( S_f : \bar{S}_f \) is of minimum capacity.

This finishes the proof of Claim 1 and the Max-Flow Min-Cut theorem.

Note also that we can construct \( S_f \) by beginning with \( S_f = \{s\} \) and then repeatedly adding any vertex \( y \notin S_f \) for which there is \( x \in S_f \) such that F1 or F2 holds. (A simple inductive argument based on sequence length shows that all of \( S_f \) is constructed in this way.)
Note also that we can construct $S_f$ by beginning with $S_f = \{s\}$ and then repeatedly adding any vertex $y \notin S_f$ for which there is $x \in S_f$ such that F1 or F2 holds.

This defines an algorithm for finding a maximum flow. The construction either finishes with $t \in S_f$ and we can augment the flow.

Or, we find that $t \notin S_f$ and we have a maximum flow.

Note, that if all the capacities $c(x, y)$ are integers and we start with the all zero flow then we find that $\theta_f$ is always a positive integer (formally one can use induction to verify this).

It follows that in this case, there is always a maximum flow that only takes integer values on the edges.
Hall’s Theorem.

Let $G = (A, B, E)$ be a bipartite graph with $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. A matching $M$ is a set of edges that meets each vertex at most once. A matching is perfect if it meets each vertex.

Hall’s theorem:

**Theorem**

$G$ contains a perfect matching iff $|N(S)| \geq |S|$ for all $S \subseteq A$.

Here $N(S) = \{b \in B : \exists a \in A \text{ s.t. } \{a, b\} \in E\}$.

Define a digraph $\Gamma$ by adding vertices $s, t \notin A \cup B$. Then add edges $(s, a_i)$ and $(b_i, t)$ of capacity 1 for $i = 1, 2, \ldots, n$. Orient the edges $E$ for $A$ to $B$ and give them capacity $\infty$. 
$G$ has a matching of size $m$ iff there is an $s - t$ flow of value $m$. An $s - t$ cut $X : \bar{X}$ has capacity

$$|A \setminus X| + |B \cap X| + |\{a \in X \cap A, b \in B \setminus X : \{a, b\} \in E\}| \times \infty.$$ 

It follows that to find a minimum cut, we need only consider $X$ such that

$$\{a \in X \cap A, b \in B \setminus X : \{a, b\} \in E\} = \emptyset. \tag{2}$$

For such a set, we let $S = A \cap X$ and $T = X \cap B$. Condition (2) means that $T \supseteq N(S)$. The capacity of $X : \bar{X}$ is now $(n - |S|) + |T|$ and for a fixed $S$ this is minimised for $T = N(S)$.

Thus, by the Max-Flow Min-Cut theorem

$$\max\{|M|\} = \min_X\{c(X : \bar{X})\} = \min_S\{n - |S| + |N(S)|\}.$$ 

This implies Hall’s theorem.
Let $G = (V, E)$ be a graph. When is it possible to orient the edges of $G$ to create a digraph $\Gamma = (V, A)$ so that every vertex has out-degree at least $d$. We say that $G$ is $d$-orientable.

**Theorem**

$G$ is $d$-orientable iff

$$\left| \{e \in E : e \cap S \neq \emptyset \} \right| \geq d|S| \text{ for all } S \subseteq V. \quad (3)$$

**Proof**

If $G$ is $d$-orientable then

$$\left| \{e \in E : e \cap S \neq \emptyset \} \right| \geq \left| \{(x, y) \in A : x \in S\} \right| \geq d|S|.$$
Suppose now that (3) holds. Define a network $N$ as follows; the vertices are $s, t, V, E$ – yes, $N$ has a vertex for each edge of $G$.

There is an edge of capacity $d$ from $s$ to each $v \in V$ and an edge of capacity one from each $e \in E$ to $t$. There is an edge of infinite capacity from $v \in V$ to each edge $e$ that contains $v$.

Consider an integer flow $f$. Suppose that $e = \{v, w\} \in E$ and $f(e, t) = 1$. Then either $f(v, e) = 1$ or $f(w, e) = 1$. In the former we interpret this as orienting the edge $e$ from $v$ to $w$ and in the latter from $w$ to $v$.

Under this interpretation, $G$ is $d$-orientable iff $N$ has a flow of value $d|V|$.
Let $X : X$ be an $s - t$ cut in $N$. Let $S = X \cap V$ and $T = X \cap E$.

To have a finite capacity, there must be no $x \in S$ and $e \in E \setminus T$ such that $x \in e$.

So, the capacity of a finite capacity cut is at least

$$d(|V| - |S|) + |\{e \in E : e \cap S \neq \emptyset\}|$$

And this is at least $d|V|$ if (3) holds.
Theorem

Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be two sets of non-negative integers where $b_1 \geq \cdots \geq b_n$. Then there is an $m \times n$ 0-1 matrix $M = (M_{i,j})$ satisfying

$$
\sum_{i=1}^{m} M_{i,j} = b_j, j \in [n] \quad \text{and} \quad \sum_{j=1}^{n} M_{i,j} \leq a_i, i \in [m] \quad (4)
$$

iff

$$
\sum_{j=1}^{k} b_j \leq \sum_{i \in A_k} a_i + k(m - |A_k|), \quad k = 0, \ldots, n - 1, \quad (5)
$$

where $A_k = \{i : a_i < k\}$

Proof  Suppose first that the matrix $M$ exists. Fix $k$ and observe that the number of 1’s in the first $k$ rows is $b_1 + \cdots + b_k$.

On the other hand the number of 1’s in the whole matrix is at least $\sum_{i \in A_k} a_i + k(m - |A_k|)$ and so (5) holds.

Now suppose that (5) holds. Define a network $N$ as follows; the vertices are $s, t, R, C$ where $R = \{r_1, \ldots, r_n\}$, $C = \{c_1, \ldots, c_n\}$.

There is an edge of capacity $b_i$ from $s$ to $r_i, i \in [n]$; an edge of capacity $a_j$ from $c_j$ to $t, j \in [n]$; an edge of capacity 1 from $r_i$ to $b_j$.

Then matrix $M$ exists if there is a flow $f$ of value $b_1 + \cdots + b_n$ from $s$ to $t$. It is defined by $M_{i,j} = f(r_i, c_j)$. 
Let $X : \bar{X}$ be an $s - t$ cut and let $S = X \cap R$, $T = X \cap C$ where $|S| = k$. The capacity of $X : \bar{X}$ is

$$\sum_{i \notin S} b_i + \sum_{j \in T} a_j + |S|(n - |T|)$$

$$\geq \sum_{i=k+1}^{n} b_i + \sum_{j \in A_k} a_j + k(n - |A|_k)$$

$$= \sum_{i=1}^{n} b_i + \left( \sum_{j \in A_k} a_j + k(n - |A|_k) - \sum_{i=1}^{k} b_i \right)$$

$$\geq \sum_{i=1}^{n} b_i,$$

as we have assume that (5) holds. Applying the Max-Flow Min-Cut theorem, we see that there is a flow of value $b_1 + \cdots + b_n$. 

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