LINEEAR ALGEBRAIC METHODS
In order to cut down the number of committees a town of $n$ people has instituted the following rules:

(a) Each club shall have an odd number of members.

(b) Each pair of clubs shall share an even number of members.

**Theorem**

*With these rules, there are at most $n$ clubs.*
Proof  Suppose that the clubs are $C_1, C_2, \ldots, C_m \subseteq [n]$.

Let $v_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})$ denote the incidence vector of $C_i$ for $1 \leq i \leq m$ i.e. $v_{i,j} = 1$ iff $j \in C_i$. We treat these vectors as being over the two element field $\mathbb{F}_2$.

We claim that $v_1, v_2, \ldots, v_m$ are linearly independent and the theorem will follow.

The rules imply that (i) $v_i \cdot v_i = 1$ and (ii) $v_i \cdot v_j = 0$ for $1 \leq i \neq j \leq m$.
(Remember that we are working over $\mathbb{F}_2$.)
Suppose then that
\[ c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0. \]

We show that \( c_1 = c_2 = \cdots = c_m = 0. \)

Indeed, we have
\[
0 = v_j \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_m v_m) \\
= c_1 v_1 \cdot v_j + c_2 v_2 \cdot v_j + \cdots + c_m v_m \cdot v_j \\
= c_j,
\]
for \( j = 1, 2, \ldots, m. \)
Point sets in $\mathbb{R}^n$ with only two distances

Let $A = \{a_1, a_2, \ldots, a_m\} \subseteq \mathbb{R}^n$.

Suppose that the pair-wise distance between elements of $A$ only take two values. How large can $A$ be?

Denote the maximum size of $A$ by $m(n)$.

**Theorem**

$$\frac{n(n+1)}{2} \leq m(n) \leq \frac{(n+1)(n+4)}{2}.$$ 

**Proof** For the lower bound we let $A = \{e_i + e_j : 1 \leq i < j \leq n\}$ where $e_i$ is the $i$th coordinate vector.
There are only two distances between elements of $A$ viz. $2^{1/2}$ and $2$.

$|A| = \frac{n(n-1)}{2}$, but it lies in the $(n-1)$-dimensional space $x_i + x_j = 2$.

For the upper bound, assume that the two distances in $A$ are $d_1, d_2$. Then consider the multivariate polynomial with $2n$ variables,

$$F(x, y) = (\|x - y\|^2 - d_1^2)(\|x - y\|^2 - d_2^2).$$

Thus our two-distance condition can be expressed:

$$F(a_i, a_j) = \begin{cases} (d_1 d_2)^2 & i = j \\ 0 & i \neq j \end{cases}$$
Next let
\[ f_i(x) = F(x, a_i) \text{ for } i = 1, 2, \ldots, m. \]

We claim that \( f_1, f_2, \ldots, f_m \) are linearly independent over \( \mathbb{R} \).
Suppose that for some \( \lambda_1, \lambda_2, \ldots, \lambda_m \)

\[ \lambda_1 f_1(x) + \lambda_2 f_2(x) + \cdots + \lambda_m f_m(x) = 0 \text{ for all } x \in \mathbb{R}^n. \]

But if \( x = a_j \) then \( f_i(x) = 0 \) for \( i \neq j \) and \( f_j(x) = (d_1 d_2)^2 \neq 0. \)

It follows that \( \lambda_j f_j(a_j) = 0 \) and the independence claim follows.
On the other hand, all polynomials $f_i$ can be expressed as linear combinations of the following:

\[
\left( \sum_{k=1}^{n} x_k^2 \right)^2, \left( \sum_{k=1}^{n} x_k^2 \right) x_j, x_i x_j, x_i, 1.
\]

The number of polynomials listed is

\[
1 + n + n(n + 1)/2 + n + 1 = (n + 1)(n + 4)/2.
\]

Thus the $f_i$ belong to a space of dimension at most $(n + 1)(n + 4)/2$. \(\square\)
Decomposing $K_n$ into bipartite subgraphs

Here we show

**Theorem**

If $G_k, k = 1, 2, \ldots, m$ is a collection of complete bipartite graphs with vertex partitions $A_k, B_k$, such that every edge of $K_n$ is in exactly one subgraph, then $m \geq n - 1$. (Note that $A_k \cap B_k = \emptyset$ here.)

**Proof**

This is tight since we can take $A_k = \{k\}, B_k = \{k + 1, \ldots, n\}$ for $k = 1, 2, \ldots, n - 1$.

Define $n \times n$ matrices $M_k$ where $M_k(i, j) = 1$ if $i \in A_k, j \in B_k$ and $M_k(i, j) = 0$ otherwise.

Let $S = M_1 + M_2 + \cdots + M_m$. Then $S + S^T = J_n - I_n$ where $I_n$ is the identity matrix and $J_n$ is the all ones matrix.

Linear algebraic methods
Decomposing $K_n$ into bipartite subgraphs

We show next that $\text{rank}(S) \geq n - 1$ and then the theorem follows from

$$\text{rank}(S) \leq \text{rank}(M_1) + \text{rank}(M_2) + \cdots + \text{rank}(M_m) \leq m.$$  

Suppose then that $\text{rank}(S) \leq n - 2$ so that there exists a non-zero solution $x = (x_1, x_2, \ldots, x_n)^T$ to the system of equations

$$Sx = 0, \quad \sum_{i=1}^{n} x_i = 0.$$  

But then, $J_nx = 0$ and $S^T x = -x$ and $-|x|^2 = -x^T S^T x = 0$, contradiction.  \qed
Theorem

Let $C_1, C_2, \ldots, C_m$ be distinct subsets of $[n]$ such that for every $i \neq j$ we have $|C_i \cap C_j| = s$ where $1 \leq s < n$. Then $m \leq n$.

Proof

If $|C_1| = s$ then $C_i \supseteq C_1$, $i = 2, 3, \ldots, m$ and the sets $C_i \setminus C_1$ are pairwise disjoint for $i \geq 2$.

It follows in this case that $m \leq 1 + n - s \leq n$.

Assume from now on that $c_i = |C_i| - s > 0$ for $i \in [m]$.
Let $M$ be the $m \times n$ 0/1 matrix where $M(i, j) = 1$ iff $j \in C_i$.

Let

$$A = MM^T = sJ + D$$

where $J$ is the $m \times m$ all 1’s matrix and $D$ is the diagonal matrix, where $D(i, i) = c_i$.

We show that $A$ and hence $M$ has rank $m$, implying that $m \leq n$ as claimed.

We will in fact show that $x^T Ax > 0$ for all $0 \neq x \in \mathbb{R}^m$. This means that $Ax \neq 0$ when $x \neq 0$. 
Nonuniform Fisher Inequality

If $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ then

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = s(x_1 + x_2 + \cdots + x_m)^2 + \sum_{i=1}^{m} c_i x_i^2 > 0.$$