PIGEON HOLE PRINCIPLE

If $f : [m] \to [n]$ then there exists $i \in [n]$ such that

$$|f^{-1}(i)| \geq \lceil m/n \rceil.$$

Informally: If m pigeons are to be placed in n pigeon-holes, at least one hole will end up with at leat $\lceil m/n \rceil$ pigeons.

Example 1: There are at least 10^4 people in China who have exactly the same number of strands of hair.

Proof. A human may have $0 \le x \le 10^5 - 1$ hair strands. There are more then 10^9 people living in China. Label a 'hole' by the number of hair strands and put a person in hole i if she/he has exactly i hair strands. There must be at least one hole with 10^4 people or more people. □

Positive integers n and k are co-prime if their largest common divisor is 1.

Example 2. If we take an arbitrary subset A of $n+1$ integers from the set $[2n] = \{1, \ldots 2n\}$ it will contain a pair of co-prime integers.

If we take the n even integers between 1 and $2n$. This set of n elements does not contain a pair of mutually prime integers. Thus we cannot replace the $n+1$ by n in the statement. We say that the statement is *tight*.

Define the holes as sets $\{1, 2\}, \{3, 4\}, \ldots \{2n-1, 2n\}$. Thus n holes are defined.

If we place the $n+1$ integers of A into their corresponding holes – by the pigeon-hole principle – there will be a hole, which will contains two numbers.

This means, that A has to contain two consecutive integers, say, x and $x+1$. But two such numbers are always co-prime.

If some integer $y \neq 1$ divides x, i.e., $x = ky$, then $x+1 = ky+1$ and this is not divisible by y. □

We have two disks, each partitioned into 200 sectors of the same size. 100 of the sectors of Disk 1 are coloured Red and 100 are colored Blue. The 200 sectors of Disk 2 are arbitrarily coloured Red and Blue.

It is always possible to place Disk 2 on top of Disk 1 so that the centres coincide, the sectors line up and at least 100 sectors of Disk 2 have the same colour as the sector underneath them.

Fix the position of Disk 1. There are 200 positions for Disk 2 and let q_i denote the number of matches if Disk 2 is placed in position i. Now for each sector of Disk 2 there are 100 positions i in which the colour of the sector underneath it coincides with its own.

Therefore

$$q_1 + q_2 + \cdots + q_{200} = 200 \times 100 \qquad (1)$$

and so there is an i such that $q_i \geq 100$.

Explanation of (1).
Consider 0-1 200×200 matrix $A(i,j)$ where $A(i,j) = 1$ iff sector j lies on top of a sector with the same colour when in position i. Row i of A has q_i 1's and column j of A has 100 1's. The LHS of (1) counts the number of 1's by adding rows and the RHS counts the number of 1's by adding columns.

Alternative solution: Place Disk 2 randomly on Disk 1 so that the sectors align. For $i = 1, 2, \dots, 200$ let

$$X_i = \begin{cases} 1 & \textit{sector } i \textit{ of disk 2 is on sector of disk 1 of same color} \\ 0 & \textit{otherwise} \end{cases}$$

We have

$$\mathbf{E}(X_i) = 1/2 \qquad \textit{for } i = 1, 2, \dots, 200.$$

So if $X = X_1 + \cdots + X_{200}$ is the number of sectors sitting above sectors of the same color, then $\mathbf{E}(X) = 100$ and there must exist at least one way to achieve 100.

Theorem

(Erdős-Szekeres) An arbitrary sequence of integers $(a_1, a_2, \ldots, a_{k^2+1})$ *contains a monotone subsequence of length* $k+1$.

Proof. Let $(a_i, a_i^1, a_i^2, \ldots, a_i^{\ell-1})$ be the longest *monotone increasing* subsequence of $(a_1, \ldots, a_{k^2+1})$ that starts with $a_i, (1 \leq i \leq k^2+1)$, and let $\ell(a_i)$ be its length.

If for some $1 \leq i \leq k^2+1$, $\ell(a_i) \geq k+1$, then $(a_i, a_i^1, a_i^2, \ldots, a_i^{l-1})$ is a monotone increasing subsequence of length $\geq k+1$.

So assume that $\ell(a_i) \leq k$ holds for every $1 \leq i \leq k^2+1$.

Consider k holes $1, 2, \ldots, k$ and place i into hole $\ell(a_i)$.

There are $k^2 + 1$ subsequences and $\leq k$ non-empty holes (different lengths), so by the pigeon-hole principle there will exist ℓ^* such that there are (at least) $k + 1$ indices $i_1, i_2, \ldots, i_{k+1}$ such that $\ell(a_{i_t}) = \ell^*$ for $1 \leq t \leq k + 1$.

Then we must have $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$.

Indeed, assume to the contrary that $a_{i_m} < a_{i_n}$ for some $1 \leq m < n \leq k + 1$. Then $a_{i_m} \leq a_{i_n} \leq a^1_{i_n} \leq a^2_{i_n} \leq \cdots \leq a^{\ell^*-1}_{i_n}$, i.e., $\ell(a_{i_m}) \geq \ell^* + 1$, a contradiction. □

The sequence

$$n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, \ldots, n^2, n^2-1, \ldots, n^2-n+1$$

has no monotone subsequence of length $n+1$ and so the Erdős-Szekerés result is best possible.

Let $P_1, P_2, \ldots, P_n$ be n points in the unit square $[0,1]^2$. We will show that there exist $i, j, k \in [n]$ such that the triangle $P_iP_jP_k$ has area

$$\leq \frac{1}{2(\lfloor\sqrt{(n-1)/2}\rfloor)^2} \sim \frac{1}{n}$$

for large n.

Let $m = \lfloor \sqrt{(n-1)/2} \rfloor$ and divide the square up into $m^2 < \frac{n}{2}$ subsquares. By the pigeonhole principle, there must be a square containing ≥ 3 points. Let 3 of these points be $P_i P_j P_k$. The area of the corresponding triangle is at most one half of the area of an individual square.