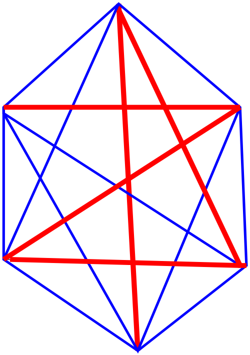
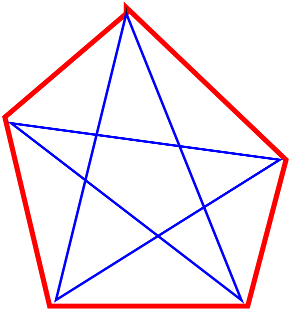
RAMSEY THEORY

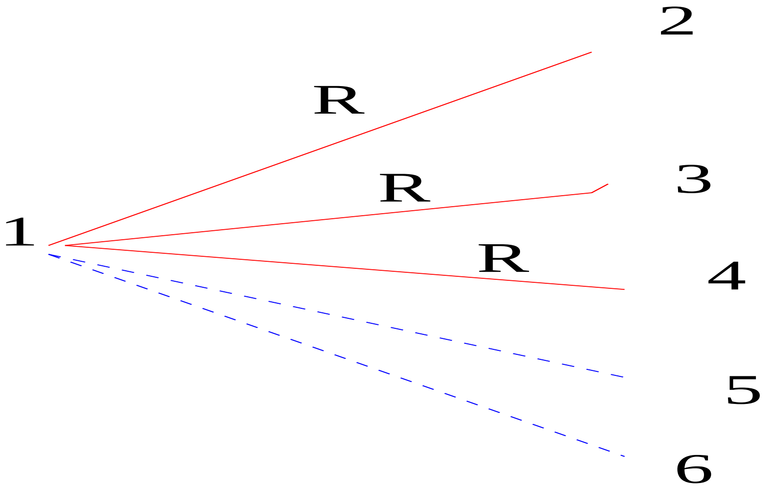
Ramsey's Theorem

Suppose we 2-colour the edges of K_6 of Red and Blue. There *must* be either a Red triangle or a Blue triangle.





This is not true for K_5.



There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter implies (1,2,3) is a Red triangle.

Ramsey's Theorem

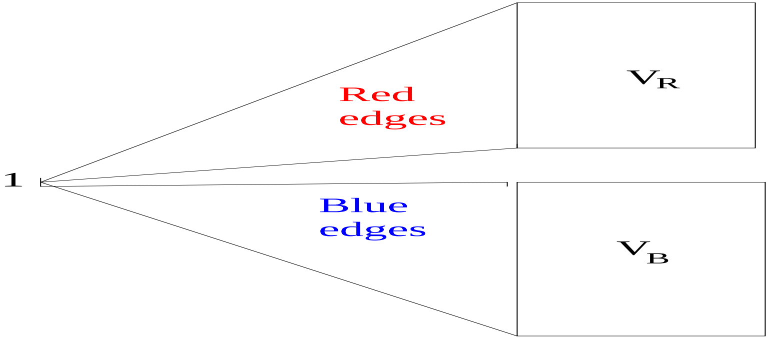
For all positive integers k, ℓ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of K_N are coloured Red or Blue then then either there is a "Red k-clique" or there is a "Blue ℓ-clique. A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

$$\begin{aligned} R(1,k) &= R(k,1) = 1 \\ R(2,k) &= R(k,2) = k \end{aligned}$$

Theorem

$$R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell).$$

Proof Let $N = R(k,\ell-1) + R(k-1,\ell)$.



$V_R = \{(x : (1,x)$ is coloured Red$\}$ and $V_B = \{(x : (1,x)$ is coloured Blue$\}$.

$$|V_R| \geq R(k-1,\ell) \text{ or } |V_B| \geq R(k,\ell-1).$$

Since

$$\begin{aligned} |V_R| + |V_B| &= N-1 \\ &= R(k,\ell-1)+R(k-1,\ell)-1. \end{aligned}$$

Suppose for example that $|V_R| \geq R(k-1,\ell)$. Then either V_R contains a Blue ℓ-clique – done, or it contains a Red $k-1$-clique K. But then $K \cup \{1\}$ is a Red k-clique. Similarly, if $|V_B| \geq R(k,\ell-1)$ then either V_B contains a Red k-clique – done, or it contains a Blue $\ell-1$-clique L and then $L \cup \{1\}$ is a Blue ℓ-clique. □

Theorem

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}.$$

Proof Induction on $k+\ell$. True for $k+\ell \leq 5$ say. Then

$$\begin{aligned} R(k,\ell) &\leq R(k,\ell-1) + R(k-1,\ell) \\ &\leq \binom{k+\ell-3}{k-1} + \binom{k+\ell-3}{k-2} \\ &= \binom{k+\ell-2}{k-1}. \end{aligned}$$

□

So, for example,

$$\begin{aligned} R(k,k) &\leq \binom{2k-2}{k-1} \\ &\leq 4^k \end{aligned}$$

Theorem

$$R(k,k) > 2^{k/2}$$

Proof We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of K_n which contains no Red k-clique and no Blue k-clique. We can assume $k \geq 4$ since we know $R(3,3) = 6$.

We show that this is true with positive probability in a *random* Red-Blue colouring. So let Ω be the set of all Red-Blue edge colourings of K_n with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: {There is a Red k-clique} and
$\mathcal{E}_B$ be the event: {There is a Blue k-clique}.
We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the N k-cliques of K_n.

Let $\mathcal{E}_{R,j}$ be the event: {C_j is Red} and let $\mathcal{E}_{B,j}$ be the event: {C_j is Blue}.

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) = 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \leq 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}
$$

Very few of the Ramsey numbers are known exactly. Here are a few known values.

$$\begin{array}{rcrcl} & & R(3,3) & = & 6 \\ & & R(3,4) & = & 9 \\ & & R(4,4) & = & 18 \\ & & R(4,5) & = & 25 \\ 43 & \leq & R(5,5) & \leq & 49 \end{array}$$

Ramsey's Theorem in general

Remember that the elements of $\binom{S}{r}$ are the r-subsets of S

Theorem

Let $r, s \geq 1$, $q_i \geq r$, $1 \leq i \leq s$ be given. Then there exists $N = N(q_1, q_2, \ldots, q_s; r)$ with the following property: Suppose that S is a set with $n \geq N$ elements. Let each of the elements of $\binom{S}{r}$ be given one of s colors. .

Then there exists i and a q_i-subset T of S such that all of the elements of $\binom{T}{r}$ are colored with the ith color.

Proof First assume that $s = 2$ i.e. two colors, Red, Blue.

Note that

$$
\begin{aligned}
(a)\; N(p,q;1) &= p+q-1 \\
(b)\; N(p,r;r) &= p(\geq r) \\
N(r,q;r) &= q(\geq r)
\end{aligned}
$$

We proceed by induction on r. It is true for $r=1$ and so assume $r \geq 2$ and it is true for $r-1$ and arbitrary p,q. Now we further proceed by induction on $p+q$. It is true for $p+q=2r$ and so assume it is true for r and all p',q' with $p'+q'<p+q$.
Let

$$
\begin{aligned}
p_1 &= N(p-1,q;r) \\
p_2 &= N(p,q-1;r)
\end{aligned}
$$

These exist by induction.

Now we prove that

$$N(p, q; r) \leq 1 + N(p_1, q_1; r-1)$$

where the RHS exists by induction.

Suppose that $n \geq 1 + N(p_1, q_1; r-1)$ and we color $\binom{[n]}{r}$ with 2 colors. Call this coloring σ.

From this we define a coloring τ of $\binom{[n-1]}{r-1}$ as follows: If $X \in \binom{[n-1]}{r-1}$ then give it the color of $X \cup \{n\}$ under σ.

Now either (i) there exists $A \subseteq [n-1]$, $|A| = p_1$ such that (under τ) all members of $\binom{A}{r-1}$ are Red or (ii) there exists $B \subseteq [n-1]$, $|A| = q_1$ such that (under τ) all members of $\binom{B}{r-1}$ are Blue.

Assume w.l.o.g. that (i) holds.

$$|A| = p_1 = N(p-1, q; r).$$

Then either

(a) $\exists B \subseteq A$ such that $|B| = q$ and under σ all of $\binom{B}{r}$ is Blue,

or

(b) $\exists A' \subseteq A$ such that $|A'| = p-1$ and all of $\binom{A'}{r}$ is Red. But then all of $\binom{A' \cup \{n\}}{r}$ is Red. If $X \subseteq A'$, $|X| = r-1$ then τ colors X Red, since $A' \subseteq A$. But then σ will color $X \cup \{n\}$ Red.

Now consider the case of s colors. We show that

$$N(q_1, q_2, \ldots, q_s; r) \leq N(Q_1, Q_2; r)$$

where

$$\begin{aligned} Q_1 &= N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r) \\ Q_2 &= N(q_{\lfloor s/2 \rfloor+1}, q_{\lfloor s/2 \rfloor+2}, \ldots, q_s; r) \end{aligned}$$

Let $n = N(Q_1, Q_2; r)$ and assume we are given an s-coloring of $\binom{[n]}{r}$.

First temporarily re-color Red, any r-set colored with $i \leq \lfloor s/2 \rfloor$ and re-color Blue any r-set colored with $i > \lfloor s/2 \rfloor$.

Then either (a) there exists a Q_1-subset A of $[n]$ with $\binom{A}{r}$ colored Red or (b) there exists a Q_2-subset B of $[n]$ with $\binom{A}{r}$ colored Blue.

W.l.o.g. assume the first case. Now replace the colors of the r-sets of A by there original colors. We have a $\lfloor s/2 \rfloor$-coloring of $\binom{A}{r}$. Since $|A| = N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r)$ there must exist some $i \leq \lfloor s/2 \rfloor$ and a q_i-subset S of A such that all of $\binom{S}{r}$ has color i. $\square$

Schur's Theorem

Let $r_k = N(3, 3, \ldots, 3; 2)$ be the smallest n such that if we k-color the edges of K_n then there is a mono-chromatic triangle.

Theorem

For all partitions $S_1, S_2, \ldots, S_k$ of $[r_k]$, there exist i and $x, y, z \in S_i$ such that $x + y = z$.

Proof Given a partition $S_1, S_2, \ldots, S_k$ of $[n]$ where $n \geq r_k$ we define a coloring of the edges of K_n by coloring (u, v) with color j where $|u - v| \in S_j$.

There will be a mono-chromatic triangle i.e. there exist j and $x < y < z$ such that $u = y - x$, $v = z - x$, $w = z - y \in S_j$. But $u + v = w$. □

A set of points X in the plane is in general position if no 3 points of X are collinear.

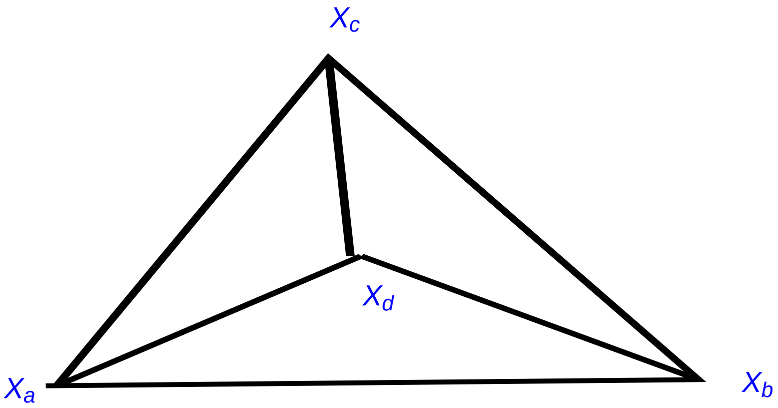
Theorem

If $n \geq N(k,k;3)$ and X is a set of n points in the plane which are in general position then X contains a k-subset Y which form the vertices of a convex polygon.

Proof We first observe that if every 4-subset of $Y \subseteq X$ forms a convex quadrilateral then Y itself induces a convex polygon.

Now label the points in S from X_1 to X_n and then color each triangle $T = \{X_i, X_j, X_k\}$, $i < j < k$ as follows: If traversing triangle $X_iX_jX_k$ in this order goes round it clockwise, color T Red, otherwise color T Blue.

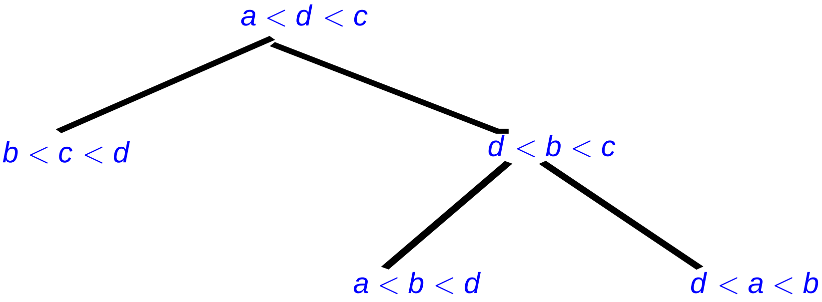
Now there must exist a k-set T such that all triangles formed from T have the same color. All we have to show is that T does not contain the following configuration:



Assume w.l.o.g. that $a < b < c$ which implies that $X_i X_j X_k$ is colored Blue.

All triangles in the previous picture are colored Blue.

So the possibilities are



and all are impossible.