



BASIC COUNTING

Let $\phi(m, n)$ be the number of mappings from $[n]$ to $[m]$.

Theorem

$$\phi(m, n) = m^n$$

Proof By induction on n .

$$\phi(m, 0) = 1 = m^0.$$

$$\begin{aligned}\phi(m, n+1) &= m\phi(m, n) \\ &= m \times m^n \\ &= m^{n+1}.\end{aligned}$$



$\phi(m, n)$ is also the number of sequences $x_1 x_2 \cdots x_n$ where $x_i \in [m]$, $i = 1, 2, \dots, n$.

Let $\psi(n)$ be the number of subsets of $[n]$.

Theorem

$$\psi(n) = 2^n.$$

Proof (1) By induction on n .

$$\psi(0) = 1 = 2^0.$$

$$\psi(n+1)$$

$$= \#\{\text{sets containing } n+1\} + \#\{\text{sets not containing } n+1\}$$

$$= \psi(n) + \psi(n)$$

$$= 2^n + 2^n$$

$$= 2^{n+1}.$$

There is a general principle that if there is a 1-1 correspondence between two finite sets A, B then $|A| = |B|$. Here is a use of this principle.

Proof (2).

For $A \subseteq [n]$ define the map $f_A : [n] \rightarrow \{0, 1\}$ by

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

f_A is the characteristic function of A .

Distinct A 's give rise to distinct f_A 's and vice-versa.

Thus $\psi(n)$ is the number of choices for f_A , which is 2^n by Theorem 1. □

Let $\psi_{\text{odd}}(n)$ be the number of odd subsets of $[n]$ and let $\psi_{\text{even}}(n)$ be the number of even subsets.

Theorem

$$\psi_{\text{odd}}(n) = \psi_{\text{even}}(n) = 2^{n-1}.$$

Proof For $A \subseteq [n-1]$ define

$$A' = \begin{cases} A & |A| \text{ is odd} \\ A \cup \{n\} & |A| \text{ is even} \end{cases}$$

The map $A \rightarrow A'$ defines a bijection between $[n-1]$ and the odd subsets of $[n]$. So $2^{n-1} = \psi(n-1) = \psi_{\text{odd}}(n)$. Furthermore,

$$\psi_{\text{even}}(n) = \psi(n) - \psi_{\text{odd}}(n) = 2^n - 2^{n-1} = 2^{n-1}.$$

Let $\phi_{1-1}(m, n)$ be the number of 1-1 mappings from $[n]$ to $[m]$.

Theorem

$$\phi_{1-1}(m, n) = \prod_{i=0}^{n-1} (m - i). \quad (1)$$

Proof Denote the RHS of (1) by $\pi(m, n)$. If $m < n$ then $\phi_{1-1}(m, n) = \pi(m, n) = 0$. So assume that $m \geq n$. Now we use induction on n .

If $n = 0$ then we have $\phi_{1-1}(m, 0) = \pi(m, 0) = 1$.

In general, if $n < m$ then

$$\begin{aligned} \phi_{1-1}(m, n+1) &= (m-n)\phi_{1-1}(m, n) \\ &= (m-n)\pi(m, n) \\ &= \pi(m, n+1). \end{aligned}$$

$\phi_{1-1}(m, n)$ also counts the number of length n **ordered** sequences **distinct** elements taken from a set of size m .

$$\phi_{1-1}(n, n) = n(n-1) \cdots 1 = n!$$

is the number of ordered sequences of $[n]$ i.e. the number of **permutations** of $[n]$.

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$$

Let X be a finite set and let

$\binom{X}{k}$ denote the collection of k -subsets of X .

Theorem

$$\left| \binom{X}{k} \right| = \binom{|X|}{k}.$$

Proof Let $n = |X|$,

$$k! \left| \binom{X}{k} \right| = \phi_{1-1}(n, k) = n(n-1)\cdots(n-k+1).$$

Let m, n be non-negative integers. Let \mathbb{Z}_+ denote the non-negative integers. Let

$$S(m, n) = \{(i_1, i_2, \dots, i_n) \in \mathbb{Z}_+^n : i_1 + i_2 + \dots + i_n = m\}.$$

Theorem

$$|S(m, n)| = \binom{m+n-1}{n-1}.$$

Proof imagine $m + n - 1$ points in a line. Choose positions $p_1 < p_2 < \dots < p_{n-1}$ and color these points red. Let $p_0 = 0, p_n = m + 1$. The gap sizes between the red points

$$i_t = p_t - p_{t-1} - 1, \quad t = 1, 2, \dots, n$$

form a sequence in $S(m, n)$ and vice-versa. □

$|S(m, n)|$ is also the number of ways of coloring m indistinguishable balls using n colors.

Suppose that we want to count the number of ways of coloring these balls so that each color appears at least once i.e. to compute $|S(m, n)^*|$ where, if $N = \{1, 2, \dots, \}$

$$S(m, n)^* =$$

$$\{(i_1, i_2, \dots, i_n) \in N^n : i_1 + i_2 + \dots + i_n = m\}$$

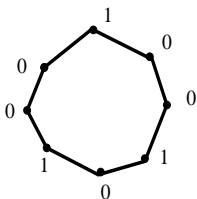
$$= \{(i_1 - 1, i_2 - 1, \dots, i_n - 1) \in Z_+^n :$$

$$(i_1 - 1) + (i_2 - 1) + \dots + (i_n - 1) = m - n\}$$

Thus,

$$|S(m, n)^*| = \binom{m - n + n - 1}{n - 1} = \binom{m - 1}{n - 1}.$$

Separated 1's on a cycle



How many ways (patterns) are there of placing k 1's and $n - k$ 0's at the vertices of a polygon with n vertices so that no two 1's are adjacent?

Choose a vertex v of the polygon in n ways and then place a 1 there. For the remainder we must choose $a_1, \dots, a_k \geq 1$ such that $a_1 + \dots + a_k = n - k$ and then go round the cycle (clockwise) putting a_1 0's followed by a 1 and then a_2 0's followed by a 1 etc..

Each pattern π arises k times in this way. There are k choices of v that correspond to a 1 of the pattern. Having chosen v there is a unique choice of a_1, a_2, \dots, a_k that will now give π .

There are $\binom{n-k-1}{k-1}$ ways of choosing the a_i and so the answer to our question is

$$\frac{n}{k} \binom{n-k-1}{k-1}$$

Theorem

Symmetry

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof Choosing r elements to include is equivalent to choosing $n - r$ elements to exclude. □

Theorem

Pascal's Triangle

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof A $k+1$ -subset of $[n+1]$ either
(i) includes $n+1$ — $\binom{n}{k}$ choices or
(ii) does not include $n+1$ — $\binom{n}{k+1}$ choices.

Theorem

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \quad (2)$$

Proof 1: Induction on n for arbitrary k .

Base case: $n = k$; $\binom{k}{k} = \binom{k+1}{k+1}$

Inductive Step: assume true for $n \geq k$.

$$\begin{aligned} \sum_{m=k}^{n+1} \binom{m}{k} &= \sum_{m=k}^n \binom{m}{k} + \binom{n+1}{k} \\ &= \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{Induction} \\ &= \binom{n+2}{k+1}. \quad \text{Pascal's triangle} \end{aligned}$$

Proof 2: Combinatorial argument.

If S denotes the set of $k + 1$ -subsets of $[n + 1]$ and S_m is the set of $k + 1$ -subsets of $[n + 1]$ which have largest element $m + 1$ then

- S_k, S_{k+1}, \dots, S_n is a partition of S .
- $|S_k| + |S_{k+1}| + \dots + |S_n| = |S|$.
- $|S_m| = \binom{m}{k}$.



Theorem

Vandermonde's Identity

$$\sum_{r=0}^k \binom{m}{r} \binom{n}{k-r} = \binom{m+n}{k}.$$

Proof Split $[m+n]$ into $A = [m]$ and $B = [m+n] \setminus [m]$. Let \mathcal{S} denote the set of k -subsets of $[m+n]$ and let $\mathcal{S}_r = \{X \in \mathcal{S} : |X \cap A| = r\}$. Then

- $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_k$ is a partition of \mathcal{S} .
- $|\mathcal{S}_0| + |\mathcal{S}_1| + \dots + |\mathcal{S}_k| = |\mathcal{S}|$.
- $|\mathcal{S}_r| = \binom{m}{r} \binom{n}{k-r}$.
- $|\mathcal{S}| = \binom{m+n}{k}$.



Theorem

Binomial Theorem

$$(1 + x)^n = \sum_{r=0}^n \binom{n}{r} x^r.$$

Proof Coefficient x^r in $(1 + x)(1 + x) \cdots (1 + x)$: choose x from r brackets and 1 from the rest. \square

Applications of Binomial Theorem

- $x = 1$:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n.$$

LHS counts the number of subsets of all sizes in $[n]$.

- $x = -1$:

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = (1 - 1)^n = 0,$$

i.e.

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

and number of subsets of even cardinality = number of subsets of odd cardinality.

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}.$$

Differentiate both sides of the Binomial Theorem w.r.t. x .

$$n(1+x)^{n-1} = \sum_{k=0}^n k \binom{n}{k} x^{k-1}.$$

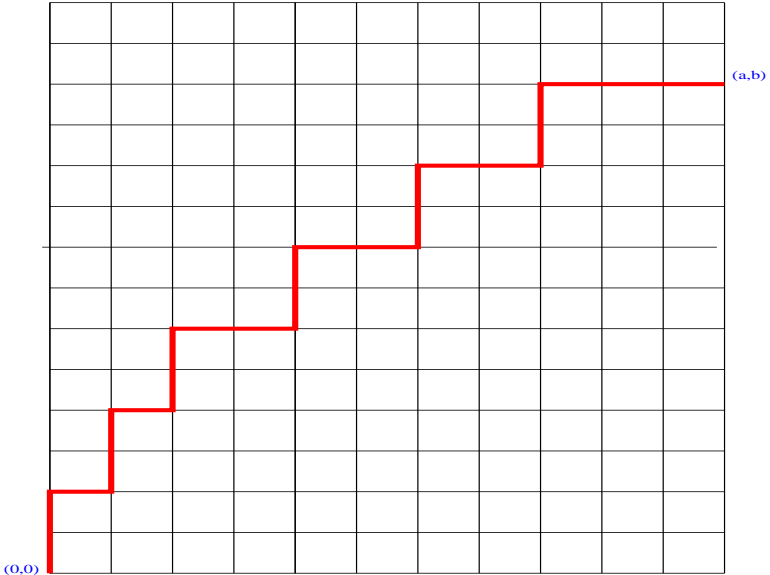
Now put $x = 1$.

Grid path problems

A *monotone path* is made up of segments $(x, y) \rightarrow (x + 1, y)$ or $(x, y) \rightarrow (x, y + 1)$.

$(a, b) \rightarrow (c, d) = \{\text{monotone paths from } (a, b) \text{ to } (c, d)\}$.

We drop the $(a, b) \rightarrow$ for paths starting at $(0, 0)$.



We consider 3 questions: Assume $a, b \geq 0$.

1. How large is $PATHS(a, b)$?

2. Assume $a < b$. Let $PATHS_{>}(a, b)$ be the set of paths in $PATHS(a, b)$ which do not touch the line $x = y$ except at $(0, 0)$. How large is $PATHS_{>}(a, b)$?

3. Assume $a \leq b$. Let $PATHS_{\geq}(a, b)$ be the set of paths in $PATHS(a, b)$ which do not pass through points with $x > y$. How large is $PATHS_{\geq}(a, b)$?

1. $STRINGS(a, b) = \{x \in \{R, U\}^* : x \text{ has } a \text{ } R\text{'s and } b \text{ } U\text{'s}\}$.¹

There is a natural bijection between $PATHS(a, b)$ and $STRINGS(a, b)$:

Path moves to Right, add R to sequence.

Path goes up, add U to sequence.

So

$$|PATHS(a, b)| = |STRINGS(a, b)| = \binom{a+b}{a}$$

since to define a string we have state which of the $a + b$ places contains an R .

¹ $\{R, U\}^*$ = set of strings of R 's and U 's

2. Every path in $\text{PATHS}_{>}(a, b)$ goes through $(0, 1)$. So

$$|\text{PATHS}_{>}(a, b)| = |\text{PATHS}((0, 1) \rightarrow (a, b))| - |\text{PATHS}_{\neq}((0, 1) \rightarrow (a, b))|.$$

Now

$$|\text{PATHS}((0, 1) \rightarrow (a, b))| = \binom{a+b-1}{a}$$

and

$$|\text{PATHS}_{\neq}((0, 1) \rightarrow (a, b))| = |\text{PATHS}((1, 0) \rightarrow (a, b))| = \binom{a+b-1}{a-1}.$$

We explain the first equality momentarily. Thus

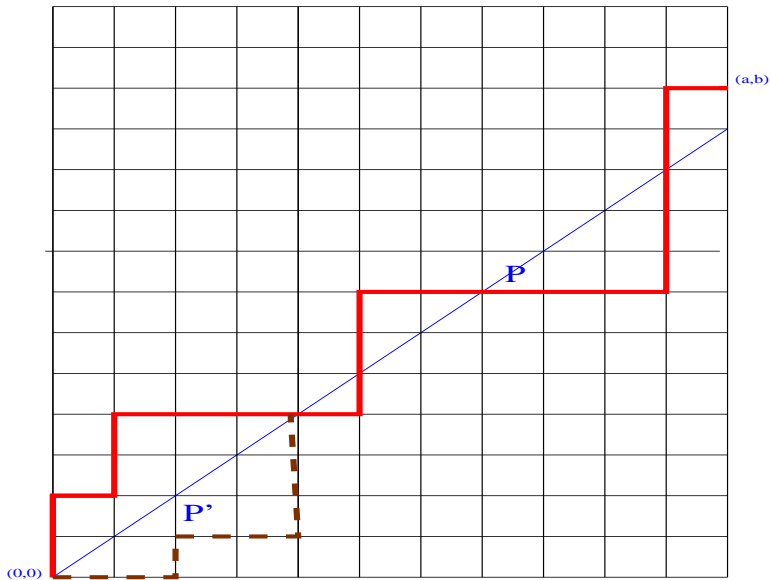
$$\begin{aligned} |\text{PATHS}_{>}(a, b)| &= \binom{a+b-1}{a} - \binom{a+b-1}{a-1} \\ &= \frac{b-a}{a+b} \binom{a+b}{a}. \end{aligned}$$

Suppose $P \in \text{PATHS}_{\neq}((0, 1) \rightarrow (a, b))$. We define $P' \in \text{PATHS}((1, 0) \rightarrow (a, b))$ in such a way that $P \rightarrow P'$ is a bijection.

Let (c, c) be the first point of P , which lies on the line $L = \{x = y\}$ and let S denote the initial segment of P going from $(0, 1)$ to (c, c) .

P' is obtained from P by deleting S and replacing it by its reflection S' in L .

To show that this defines a bijection, observe that if $P' \in \text{PATHS}((1, 0) \rightarrow (a, b))$ then a similarly defined *reverse reflection* yields a $P \in \text{PATHS}_{\neq}((0, 1) \rightarrow (a, b))$.



3. Suppose $P \in \text{PATHS}_{\geq}(a, b)$. We define $P' \in \text{PATHS}_{>}(a, b+1)$ in such a way that $P \rightarrow P'$ is a bijection.

Thus

$$|\text{PATHS}_{\geq}(a, b)| = \frac{b-a+1}{a+b+1} \binom{a+b+1}{a}.$$

In particular

$$\begin{aligned} |\text{PATHS}_{\geq}(a, a)| &= \frac{1}{2a+1} \binom{2a+1}{a} \\ &= \frac{1}{a+1} \binom{2a}{a}. \end{aligned}$$

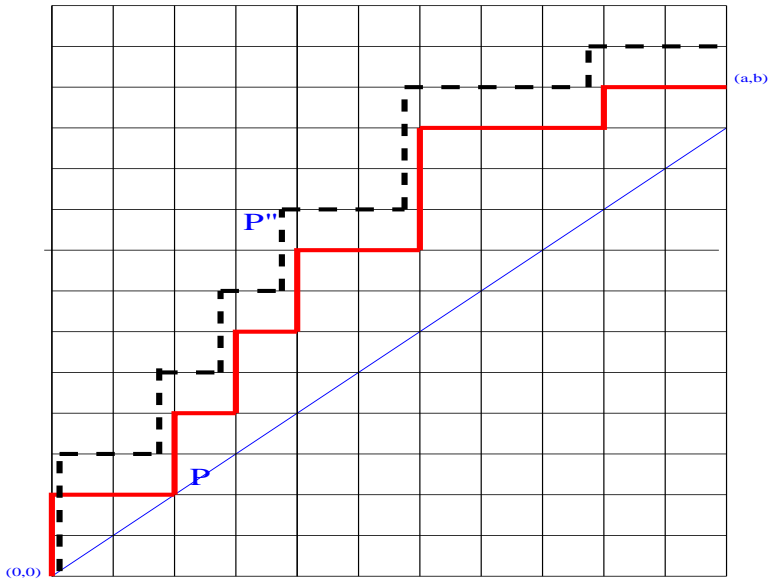
The final expression is called a *Catalan Number*.

The bijection

Given P we obtain P'' by *raising it vertically one position and then adding the segment $(0, 0) \rightarrow (0, 1)$* .

More precisely, if $P = (0, 0), (x_1, y_1), (x_2, y_2), \dots, (a, b)$ then $P'' = (0, 0), (0, 1), (x_1, y_1 + 1), \dots, (a, b + 1)$.

This is clearly a $1 - 1$ onto function between $\text{PATHS}_{\geq}(a, b)$ and $\text{PATHS}_{>}(a, b + 1)$.



Multi-sets

Suppose we allow elements to appear several times in a set:

$\{a, a, a, b, b, c, c, c, d, d\}$.

To avoid confusion with the standard definition of a set we write

$\{3 \times a, 2 \times b, 3 \times c, 2 \times d\}$.

How many distinct permutations are there of the multiset

$\{a_1 \times 1, a_2 \times 2, \dots, a_n \times n\}$?

Ex. $\{2 \times a, 3 \times b\}$.

aabbb; ababb; abbab; abbba; baabb

babab; babba; bbaab; bbaba; bbbaa.

Start with $\{a_1, a_2, b_1, b_2, b_3\}$ which has $5! = 120$ permutations:
 $\dots a_2 b_3 a_1 b_2 b_1 \dots a_1 b_2 a_2 b_1 b_3 \dots$

After erasing the subscripts each possible sequence e.g.

$ababb$ occurs $2! \times 3!$ times and so the number of permutations is $5!/2!3! = 10$.

In general if $m = a_1 + a_2 + \dots + a_n$ then the number of permutations is

$$\frac{m!}{a_1! a_2! \dots a_n!}$$

Multinomial Coefficients

$$\binom{m}{a_1, a_2, \dots, a_n} = \frac{m!}{a_1! a_2! \cdots a_n!}$$

$$(x_1 + x_2 + \cdots + x_n)^m =$$

$$\sum_{\substack{a_1 + a_2 + \cdots + a_n = m \\ a_1 \geq 0, \dots, a_n \geq 0}} \binom{m}{a_1, a_2, \dots, a_n} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

E.g.

$$\begin{aligned} (x_1 + x_2 + x_3)^4 &= \binom{4}{4, 0, 0} x_1^4 + \binom{4}{3, 1, 0} x_1^3 x_2 + \\ &\quad \binom{4}{3, 0, 1} x_1^3 x_3 + \binom{4}{2, 1, 1} x_1^2 x_2 x_3 + \cdots \\ &= x_1^4 + 4x_1^3 x_2 + 4x_1^3 x_3 + 12x_1^2 x_2 x_3 + \cdots \end{aligned}$$

Contribution of 1 to the coefficient of $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ from every permutation in $S = \{x_1 \times a_1, x_2 \times a_2, \dots, x_n \times a_n\}$.

E.g.

$$(x_1 + x_2 + x_3)^6 = \dots + x_2 x_3 x_2 x_1 x_1 x_3 + \dots$$

where the displayed term comes by choosing x_2 from first bracket, x_3 from second bracket etc.

Given a permutation $i_1 i_2 \dots i_m$ of S e.g. 331422 \dots we choose x_3 from the first 2 brackets, x_1 from the 3rd bracket etc. Conversely, given a choice from each bracket which contributes to the coefficient of $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ we get a permutation of S .

Balls in boxes

m distinguishable balls are placed in n distinguishable boxes.
Box i gets b_i balls.

$$\# \text{ ways is } \binom{m}{b_1, b_2, \dots, b_n}.$$

$$m = 7, n = 3, b_1 = 2, b_2 = 2, b_3 = 3$$

No. of ways is

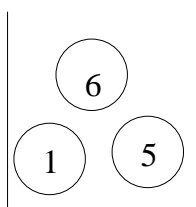
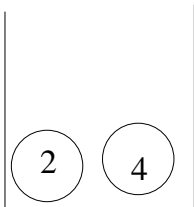
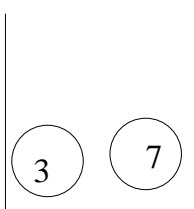
$$7!/(2!2!3!) = 210$$

$$[1, 2][3, 4][5, 6, 7] \quad [1, 2][3, 5][4, 6, 7] \quad \dots \quad [6, 7][4, 5][1, 2, 3]$$

3 1 3 2 1 3 2

Ball 1 goes in box 3, Ball 2 goes in box 1, etc.

Conversely, given an allocation of balls to boxes:



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Inclusion-Exclusion

2 sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

So if $A_1, A_2 \subseteq A$ and $\bar{A}_i = A \setminus A_i$, $i = 1, 2$ then

$$|\bar{A}_1 \cap \bar{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3 sets:

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= |A| - |A_1| - |A_2| - |A_3| \\ &\quad + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

General Case

$$A_1, A_2, \dots, A_N \subseteq A.$$

$$\text{For } S \subseteq [N], A_S = \bigcap_{i \in S} A_i.$$

$$\text{E.g. } A_{\{4,7,18\}} = A_4 \cap A_7 \cap A_{18}.$$

$$A_\emptyset = A.$$

Inclusion-Exclusion Formula:

$$\left| \bigcap_{i=1}^N \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.$$

Simple example. How many integers in $[1000]$ are not divisible by 5,6 or 8 i.e. what is the size of $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ below?

$A = A_\emptyset$	$= \{1, 2, 3, \dots, \}$	$ A = 1000$
A_1	$= \{5, 10, 15, \dots, \}$	$ A_1 = 200$
A_2	$= \{6, 12, 18, \dots, \}$	$ A_2 = 166$
A_3	$= \{8, 16, 24, \dots, \}$	$ A_3 = 125$
$A_{\{1,2\}}$	$= \{30, 60, 90, \dots, \}$	$ A_{\{1,2\}} = 33$
$A_{\{1,3\}}$	$= \{40, 80, 120, \dots, \}$	$ A_{\{1,3\}} = 25$
$A_{\{2,3\}}$	$= \{24, 48, 72, \dots, \}$	$ A_{\{2,3\}} = 41$
$A_{\{1,2,3\}}$	$= \{120, 240, 360, \dots, \}$	$ A_{\{1,2,3\}} = 8$

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= 1000 - (200 + 166 + 125) \\ &\quad + (33 + 25 + 41) - 8 \\ &= 600. \end{aligned}$$

Derangements

A **derangement** of $[n]$ is a permutation π such that

$$\pi(i) \neq i : i = 1, 2, \dots, n.$$

We must express the set of derangements D_n of $[n]$ as the intersection of the complements of sets.

We let $A_i = \{\text{permutations } \pi : \pi(i) = i\}$ and then

$$|D_n| = \left| \bigcap_{i=1}^n \overline{A_i} \right|.$$

We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n - 1)!$: after fixing $\pi(1) = 1$ there are $(n - 1)!$ ways of permuting $2, 3, \dots, n$.

$|A_{\{1,2\}}| = (n - 2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n - 2)!$ ways of permuting $3, 4, \dots, n$.

In general

$$|A_S| = (n - |S|)!$$

$$\begin{aligned} |D_n| &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} (n - k)! \\ &= \sum_{k=0}^n (-1)^k \frac{n!}{k!} \\ &= n! \sum_{k=0}^n (-1)^k \frac{1}{k!}. \end{aligned}$$

Proof of inclusion-exclusion formula

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$$(1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^N \bar{A}_i \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned} \left| \bigcap_{i=1}^N \bar{A}_i \right| &= \sum_{x \in A} (1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) \\ &= \sum_{x \in A} \sum_{S \subseteq [N]} (-1)^{|S|} \prod_{i \in S} \theta_{x,i} \\ &= \sum_{S \subseteq [N]} (-1)^{|S|} \sum_{x \in A} \prod_{i \in S} \theta_{x,i} \\ &= \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|. \end{aligned}$$

Euler's Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n , other than 1.

$$\phi(12) = 4.$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of n .

$$A_i = \{x \in [n] : p_i \text{ divides } x\}, \quad 1 \leq i \leq k.$$

$$\phi(n) = \left| \bigcap_{i=1}^k \bar{A}_i \right|$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i} \quad S \subseteq [k].$$

$$\begin{aligned} \phi(n) &= \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} \\ &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \end{aligned}$$

Surjections

Fix n, m . Let

$$A = \{f : [n] \rightarrow [m]\}$$

Thus $|A| = m^n$. Let

$$F(n, m) = \{f \in A : f \text{ is onto } [m]\}.$$

How big is $F(n, m)$?

Let

$$A_i = \{f \in F : f(x) \neq i, \forall x \in [n]\}.$$

Then

$$F(n, m) = \bigcap_{i=1}^m \bar{A}_i.$$

For $S \subseteq [m]$

$$\begin{aligned}A_S &= \{f \in A : f(x) \notin S, \forall x \in [n]\}. \\ &= \{f : [n] \rightarrow [m] \setminus S\}.\end{aligned}$$

So

$$|A_S| = (m - |S|)^n.$$

Hence

$$\begin{aligned}F(n, m) &= \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n \\ &= \sum_{k=0}^m (-1)^k \binom{m}{k} (m - k)^n.\end{aligned}$$

Scrambled Allocations

We have n boxes B_1, B_2, \dots, B_n and $2n$ distinguishable balls b_1, b_2, \dots, b_{2n} .

An allocation of balls to boxes, **two balls to a box**, is said to be *scrambled* if there does **not** exist i such that box B_i contains balls b_{2i-1}, b_{2i} . Let σ_n be the number of scrambled allocations.

Let A_i be the set of allocations in which box B_i contains b_{2i-1}, b_{2i} . We show that

$$|A_S| = \frac{(2(n - |S|))!}{2^{n-|S|}}.$$

Inclusion-Exclusion then gives

$$\sigma_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(2(n-k))!}{2^{n-k}}.$$

First consider A_\emptyset :

Each permutation π of $[2n]$ yields an allocation of balls, placing $b_{\pi(2i-1)}, b_{\pi(2i)}$ into box B_i , for $i = 1, 2, \dots, n$. The order of balls in the boxes is immaterial and so each allocation comes from exactly 2^n distinct permutations, giving

$$|A_\emptyset| = \frac{(2n)!}{2^n}.$$

To get the formula for $|A_S|$ observe that the contents of $2|S|$ boxes are fixed and so we are in essence dealing with $n - |S|$ boxes and $2(n - |S|)$ balls.

Problème des Ménages

In how many ways M_n can n male-female couples be seated around a table, alternating male-female, so that no person is seated next to their partner?

Let A_i be the set of seatings in which couple i sit together.

If $|S| = k$ then

$$|A_S| = 2k!(n - k)!^2 \times d_k.$$

d_k is the number of ways of placing k 1's on a cycle of length $2n$ so that no two 1's are adjacent. (We place a person at each 1 and his/her partner on the succeeding 0).

2 choices for which seats are occupied by the men or women.

$k!$ ways of assigning the couples to the positions; $(n - k)!^2$

ways of assigning the rest of the people.

$$d_k = \frac{2n}{k} \binom{2n-k-1}{k-1} = \frac{2n}{2n-k} \binom{2n-k}{k}.$$

(See slides 11 and 12).

$$\begin{aligned} M_n &= \sum_{k=0}^n (-1)^k \binom{n}{k} \times 2k!(n-k)!^2 \times \frac{2n}{2n-k} \binom{2n-k}{k} \\ &= 2n! \sum_{k=0}^n (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!. \end{aligned}$$