GENERATING FUNCTIONS AND RECURRENCE RELATIONS

Recurrence Relations

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots,$is an infinite sequence.
A recurrence recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \tag{1}$$

The whole sequence is determined by (1) and the values of $a_0, a_1, \ldots, a_{k-1}$.

Linear Recurrence

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2.$$

$a_0 = a_1 = 1$.

$b_n = |B_n| = |\{x \in \{a, b, c\}^n : aa \text{ does not occur in } x\}|.$

$b_1 = 3 :\ a\ b\ c$

$b_2 = 8 :\ ab\ ac\ ba\ bb\ bc\ ca\ cb\ cc$

$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$$

Let

$$B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)}$$

where $B_n^{(\alpha)} = \{x \in B_n : \ x_1 = \alpha\}$ for $\alpha = a, b, c$.

Now $|B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}|$. The map $f : B_n^{(b)} \to B_{n-1}$,

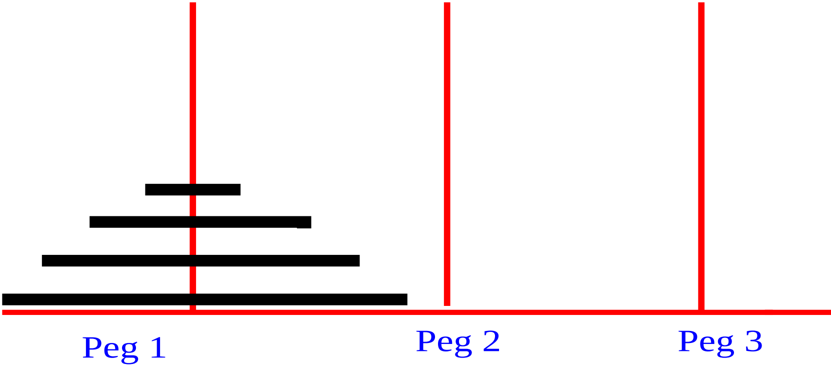
$$f(bx_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

$B_n^{(a)} = \{x \in B_n : \ x_1 = a \text{ and } x_2 = b \text{ or } c\}$. The map $g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(c)}$,

$$g(ax_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

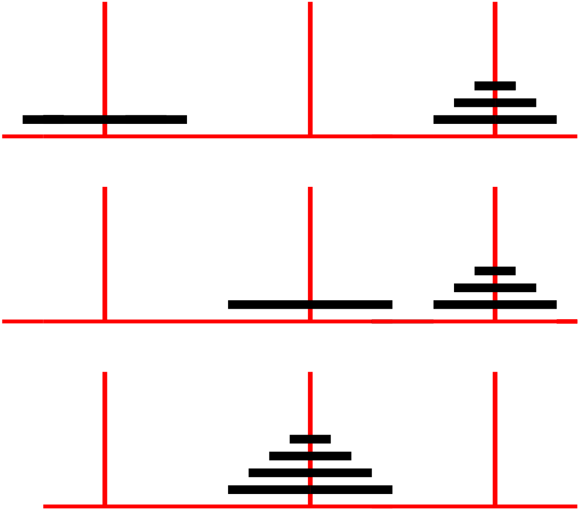
Hence, $|B_n^{(a)}| = 2|B_{n-2}|$.

Towers of Hanoi



H_n is the minimum number of moves needed to shift n rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.

$H_n = 2H_{n-1} + 1$



H_{n-1} moves

1 move

H_{n-1} moves

A has n dollars. Everyday A buys one of a Bun (1 dollar), an Ice-Cream (2 dollars) or a Pastry (2 dollars). How many ways are there (sequences) for A to spend his money?
Ex. BBPIIPBI represents “Day 1, buy Bun. Day 2, buy Bun etc.”.

$$\begin{aligned} u_n &= \text{number of ways} \\ &= u_{n,B} + u_{n,I} + u_{n,P} \end{aligned}$$

where $u_{n,B}$ is the number of ways where A buys a Bun on day 1 etc.
$u_{n,B} = u_{n-1},\ u_{n,I} = u_{n,P} = u_{n-2}$.
So

$$u_n = u_{n-1} + 2u_{n-2},$$

and

$$u_0 = u_1 = 1.$$

If $a_0, a_1, \ldots, a_n$ is a sequence of real numbers then its **(ordinary) generating function** $a(x)$ is given by

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

and we write

$$a_n = [x^n]a(x).$$

$a_n = 1$

$$a(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$a_n = n + 1$.

$$a(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$$

$a_n = n$.

$$a(x) = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$$

Generalised binomial theorem:

$a_n = \binom{\alpha}{n}$

$$a(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

$a_n = \binom{m+n-1}{n}$

$$a(x) = \frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n.$$

General view.

Given a recurrence relation for the sequence (a_n), we

(a) Deduce from it, an equation satisfied by the generating function $a(x) = \sum_n a_n x^n$.

(b) Solve this equation to get an explicit expression for the generating function.

(c) Extract the coefficient a_n of x^n from $a(x)$, by expanding $a(x)$ as a power series.

Solution of linear recurrences

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \qquad n \geq 2.$$

$a_0 = 1, a_1 = 9$.

$$\sum_{n=2}^{\infty} (a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \tag{2}$$

$$
\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= a(x) - a_0 - a_1 x \\
&= a(x) - 1 - 9x. \\
\sum_{n=2}^{\infty} 6a_{n-1} x^n &= 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \\
&= 6x(a(x) - a_0) \\
&= 6x(a(x) - 1). \\
\sum_{n=2}^{\infty} 9a_{n-2} x^n &= 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 9x^2 a(x).
\end{aligned}
$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.$$

$$\begin{aligned} a(x) &= \frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(1-3x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + 3x\sum_{n=0}^{\infty}(n+1)3^n x^n \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n \\ &= \sum_{n=0}^{\infty}(2n+1)3^n x^n. \end{aligned}$$

$$a_n = (2n+1)3^n.$$

Fibonacci sequence:

$$\sum_{n=2}^{\infty}(a_n - a_{n-1} - a_{n-2})x^n = 0.$$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(a(x) - a_0 - a_1 x) - (x(a(x) - a_0)) - x^2 a(x) = 0.$$

$$a(x) = \frac{1}{1 - x - x^2}.$$

$$\begin{aligned} a(x) &= -\frac{1}{(\xi_1 - x)(\xi_2 - x)} \\ &= \frac{1}{\xi_1 - \xi_2}\left(\frac{1}{\xi_1 - x} - \frac{1}{\xi_2 - x}\right) \\ &= \frac{1}{\xi_1 - \xi_2}\left(\frac{\xi_1^{-1}}{1 - x/\xi_1} - \frac{\xi_2^{-1}}{1 - x/\xi_2}\right) \end{aligned}$$

where

$$\xi_1 = -\frac{\sqrt{5}+1}{2} \text{ and } \xi_2 = \frac{\sqrt{5}-1}{2}$$

are the 2 roots of

$$x^2 + x - 1 = 0.$$

Therefore,

$$\begin{aligned} a(x) &= \frac{\xi_1^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_1^{-n} x^n - \frac{\xi_2^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_2^{-n} x^n \\ &= \sum_{n=0}^{\infty} \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} x^n \end{aligned}$$

and so

$$\begin{aligned} a_n &= \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right). \end{aligned}$$

Inhomogeneous problem

$$a_n - 3a_{n-1} = n^2 \qquad n \geq 1.$$

$a_0 = 1$.

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= \sum_{n=1}^{\infty} n^2 x^n \\
\sum_{n=1}^{\infty} n^2 x^n &= \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^n \\
&= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= \frac{x+x^2}{(1-x)^3} \\
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= a(x) - 1 - 3xa(x) \\
&= a(x)(1-3x) - 1.
\end{aligned}$$

$$\begin{aligned} a(x) &= \frac{x+x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x} \\ &= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D+1}{1-3x} \end{aligned}$$

where

$$\begin{aligned} x+x^2 &\cong A(1-x)^2(1-3x) + B(1-x)(1-3x) \\ &\qquad + C(1-3x) + D(1-x)^3. \end{aligned}$$

Then

$$A = -1/2,\ B = 0,\ C = -1,\ D = 3/2.$$

So

$$\begin{aligned} a(x) &= \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n \end{aligned}$$

So

$$\begin{aligned} a_n &= -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2}3^n \\ &= -\frac{3}{2} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2}3^n. \end{aligned}$$

General case of linear recurrence

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = u_n, \qquad n \geq k.$$

$u_0, u_1, \ldots, u_{k-1}$ are given.

$$\sum (a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} - u_n) x^n = 0$$

It follows that for some polynomial $r(x)$,

$$a(x) = \frac{u(x) + r(x)}{q(x)}$$

where

$$q(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_k x^k = \prod_{i=1}^{k} (1 - \alpha_i x)$$

and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of $p(x) = 0$ where $p(x) = x^k q(1/x) = x^k + c_1 x^{k-1} + \cdots + c_0$.

Products of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n, \; b(x)) = \sum_{n=0}^{\infty} b_n x^n.$$

$$\begin{aligned} a(x)b(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots) \times \\ &\qquad\qquad (b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \\ &\qquad\qquad (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Combinatorially, suppose that A, B are sets of objects and A contains a_n objects of weight n and B contains b_n objects of weight n, for all $n \geq 0$.

We now consider pairs of objects (a, b) where $a \in A, b \in B$. The weight of pair (a, b) is the weight of a plus the weight of b.

c_n is the number of pairs of weight n.

Simple example:
Let $A = B = \{0, 1, 2, \dots, \}$ where the weight of i is i.

We want to count the number of pairs (a, b) with $a + b = n$.
(This is of course $n + 1$).

$a_n = b_n = 1$, $n \geq 0$ and so $a(x) = b(x) = 1/(1 - x)$.

Thus

$$\begin{aligned} c(x) &= \frac{1}{(1-x)^2} \\ &= \sum_{n=0}^{\infty} (n+1)x^n. \end{aligned}$$

Suppose that we have r generating functions $a_1(x), a_2(x), \ldots, a_r(x)$ where

$$a_i(x) = \sum_{n=0}^{\infty} a_{i,n} x^n$$

and $a_{i,n}$ is the number of objects in A_i and weight n. Then if $c(x) = a_1(x)a_2(x)\cdots a_r(x) = \sum_{n\geq 0} c_n x^n$ then c_n is the number of sequences $x_1 x_2 \cdots x_r$ where

$$weight(x_1) + weight(x_2) + \cdots + weight(x_r) = n$$

This can be proved by induction on r.

As an example, let $A_1 = A_2 = \cdots = A_r = \{0, 1, 2, \ldots, \}$. Then the number of sequences $x_1, x_2, \ldots, x_r \geq 0$ satisfying $x_1 + \cdots x_r = n$ is given by

$$[x^n]\frac{1}{(1-x)^r} = (-1)^n \binom{-r}{n} = \binom{n+r-1}{n}.$$

Derangements

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}.$$

Explanation: $\binom{n}{k} d_{n-k}$ is the number of permutations with exactly k cycles of length 1. Choose k elements ($\binom{n}{k}$ ways) for which $\pi(i) = i$ and then choose a derangement of the remaining $n-k$ elements.

So

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \\ \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n. \end{aligned} \tag{3}$$

Let

$$d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m.$$

From (3) we have

$$\begin{aligned} \frac{1}{1-x} &= e^x d(x) \\ d(x) &= \frac{e^{-x}}{1-x} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right) x^n. \end{aligned}$$

So

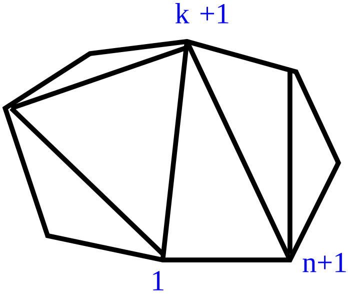
$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Triangulation of n-gon

Let

$$\begin{aligned} a_n &= \text{number of triangulations of } P_{n+1} \\ &= \sum_{k=0}^{n} a_k a_{n-k} \qquad n \geq 2 \end{aligned} \tag{4}$$

$a_0 = 0$, $a_1 = a_2 = 1$.



Explanation of (4):

$a_k a_{n-k}$ counts the number of triangulations in which edge $1, n+1$ is contained in triangle $1, k+1, n+1$.

There are a_k ways of triangulating $1, 2, \ldots, k+1, 1$ and for each such there are a_{n-k} ways of triangulating $k+1, k+2, \ldots, n+1, k+1$.

$$x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n .$$

But,

$$x + \sum_{n=2}^{\infty} a_n x^n = a(x)$$

since $a_0 = 0, a_1 = 1$.

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n \\ &= a(x)^2 . \end{aligned}$$

So

$$a(x) = x + a(x)^2$$

and hence

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \text{ or } \frac{1-\sqrt{1-4x}}{2}.$$

But $a(0) = 0$ and so

$$\begin{aligned} a(x) &= \frac{1-\sqrt{1-4x}}{2} \\ &= \frac{1}{2} - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n. \end{aligned}$$

So

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}.$$