1. Given a set of \( n^2 + 1 \) positive integers, show that either there exists a subset \( A \) of size \( n + 1 \) such that either (1) no element of \( A \) divides another element of \( A \), or (2) for every \( a, b \in A \) with \( a < b \), we have \( a \) divides \( b \).

**Solution:**
This follows directly from Dilworth’s theorem. If there is no set \( A \) satisfying (1) then the maximum size of anti-chain in the divisibility poset is at most \( n \). Therefore the poset can be covered by at most \( n \) chains. One of which must be of size at least \( n + 1 \), giving (2).

2. (a) How many strings of length \( n \) consisting of 0’s and 1’s have no two consecutive 1’s?
(b) How many strings of length \( n \) consisting of 0’s and 1’s have no three consecutive 1’s and no three consecutive 0’s?

**Solution:**
(a) Let \( \alpha_n \) be the number of strings made of zeros and ones with no two consecutive ones. If \( a_n \) ends in a 0, we have \( \alpha_{n-1} \) possible strings. If \( a_n \) ends in a 1, it must end in a 01, so we have \( \alpha_{n-2} \) possible strings. So,

\[
\alpha_n = \alpha_{n-1} + \alpha_{n-2}.
\]

There is one empty valid sequence, two valid sequences of length 1 and three of length 2.
Therefore \( \alpha_n = F_{n+1} \), where \( F_n \) is the \( n \)'th Fibonacci number.

(b) Let \( a_n \) be a string of length \( n \) that satisfies the condition in the problem. Define \( b_{n-1} \) as follows: \( b_i = 1 \) iff \( a_i = a_{i+1} \) and 0 otherwise. The string \( b_{n-1} \) has no two consecutive ones. From (a) above, there are \( F_n \) strings of the defined type. For each string \( b_{n-1} \) there are 2 strings \( a_n \). So, the answer is \( 2F_n \).

3. Find \( a_n \) if \( a_n = 6a_{n-1} + 7a_{n-2}, a_0 = 2, a_1 = 10 \).

**Solution:**
Let \( a(x) = \sum a_n x^n \). Then

\[
a(x) - 2 - 10x = 6x(a(x) - 2) + 7x^2 a(x),
\]

\[
a(x) = \frac{2 - 2x}{1 - 6x - 7x^2};
\]

\[
a(x) = \frac{3/2}{1 - 7x} + \frac{1/2}{1 + x}.
\]

So

\[
a_n = \frac{3}{2} 7^n + \frac{1}{2} (-1)^n.
\]