21-301 Combinatorics
Homework 7
Due: Friday, November 6

1. Given a set of $n^2 + 1$ positive integers, show that either there exists a subset A of size $n+1$ such that either (1) no element of A divides another element of A, or (2) for every $a, b \in A$ with $a < b$, we have a divides b.

   Solution:
   This follows directly from Dilworth's theorem. If there is no set A satisfying (1) then the maximum size of anti-chain in the divisibility poset is at most n. Therefore the poset can be covered by at most n chains. One of which must be of size at least $n + 1$, giving (2).

2. (a) How many strings of length n consisting of 0's and 1's have no two consecutive 1's?
   (b) How many strings of length n consisting of 0's and 1's have no three consecutive 1's and no three consecutive 0's?

   Solution:

   (a) Let α_n be the number of strings made of zeros and ones with no two consecutive ones. If $\mathbf{a}_n$ ends in a 0, we have α_{n-1} possible strings. If $\mathbf{a}_n$ ends in a 1, it must end in a 01, so we have α_{n-2} possible strings. So,

   $$\alpha_n = \alpha_{n-1} + \alpha_{n-2}.$$

   There is one empty velid sequence, two valid sequences of length 1 and three of length 2.
   Therefore $\alpha_n = F_{n+1}$, where F_n is the n'th Fibonacci number.

   (b) Let $\mathbf{a}_n$ be a string of length n that satisfies the condition in the problem. Define $\mathbf{b}_{n-1}$ as follows: $b_i = 1$ iff $a_i = a_{i+1}$ and 0 otherwise. The string $\mathbf{b}_{n-1}$ has no two consecutive ones. From (a) above, there are F_n strings of the defined type. For each string $\mathbf{b}_{n-1}$ there are 2 strings $\mathbf{a}_n$. So, the answer is $2F_n$.

3. Find a_n if

   $$a_n = 6a_{n-1} + 7a_{n-2}, a_0 = 2, a_1 = 10.$$

Solution:
Let $a(x) = \sum a_n x^n$. Then

$$a(x) - 2 - 10x = 6x(a(x) - 2) + 7x^2 a(x),$$

$$a(x) = \frac{2 - 2x}{1 - 6x - 7x^2},$$

$$a(x) = \frac{3/2}{1 - 7x} + \frac{1/2}{1 + x}.$$

So

$$a_n = \frac{3}{2}7^n + \frac{1}{2}(-1)^n.$$