1. Suppose that in a town of $n$ citizens, clubs $C_1, C_2, \ldots, C_m$ must satisfy (i) $|C_i|$ is even for $i = 1, 2, \ldots, m$ and (ii) $|C_i \cap C_j|$ is odd for $1 \leq i < j \leq m$. Show that $m \leq n$.

(Hint: Consider the cases $n$ odd and $n$ even separately. In the $n$ even case consider the matrix $M = AA^T$ where $A$ is made up from the incidence vectors of the columns of $C_1, C_2, \ldots, C_m$.)

**Solution:** Let $1$ denote the all ones vector of dimension $n$. Let $x_i, i = 1, 2, \ldots, m$ denote the incidence vectors of the clubs $C_1, C_2, \ldots, C_m$.

Assume first that $n$ is odd. In which case we let $D_j = [n] \setminus C_j$ for $j = 1, 2, \ldots, n$. Then $|D_j|$ is odd for $j = 1, 2, \ldots, n$ and $|D_i \cap D_j| = n - |C_i \cup C_j| = n - |C_i| - |C_j| + |C_i \cap C_j|$ is even for $i \neq j$.

The result now follows from Odd town.

The $n$ even case is more tricky. Let $A$ be the $n \times m$ matrix with columns $x_1, x_2, \ldots, x_m$.

Our assumptions imply that (i) $1^T A = 0$ and so $\text{rank}(A) \leq n - 1$ and (ii) $B = AA^T = J - I$. We now have the contradiction that $n = \text{rank}(B) \leq \text{rank}(A) \leq n - 1$, where $\text{rank}(B) = n$ follows from $B^2 = J^2 - J - J + I = I$.

2. Let $A_1, A_2, \ldots, A_m \subseteq [n]$ be such that for $1 \leq i < j \leq m$, $d_{i,j} = |(A_i \setminus A_j) \cup (A_j \setminus A_i)|$ takes one of two values. Show that $m \leq (n + 1)(n + 4)/2$.

**Solution:** Let $x_i$ be the incidence vector of $A_i$ for $i = 1, 2, \ldots, m$. Then $||x_i - x_j|| \in \left\{d_1^{1/2}, d_2^{1/2}\right\}$ where $d_1, d_2$ are the two values mentioned in the question.

3. Each edge of $K_n$ appears an odd number of times as an edge in the collection $G_1, G_2, \ldots, G_m$ of bipartite subgraphs of $K_n$. Show that $m \geq (n - 1)/2$.

[Hint: Let $A_k, B_k, M_k, S$ be as in the notes on Linear Algebraic Methods and consider the $2n \times n$ matrix $T = \left[\begin{array}{c} S \\ \ \ \ I_n \end{array}\right]$.

**Solution:** We deduce from the conditions in the question that $S + S^T = J_n - I_n$ over $F_2$.

Furthermore, if $m < (n - 1)/2$ then $\text{rank}(T) \leq n - 1$ and there exists $x \neq 0$ such that $Tx = 0$. But then, (1) implies that $x = 0$, contradiction.