21-301 Combinatorics
Homework 6
Due: Monday, October 26

1. Suppose that in a town of n citizens, clubs $C_1, C_2, \ldots, C_m$ must satisfy (i) $|C_i|$ is even for $i = 1, 2, \ldots, m$ and (ii) $|C_i \cap C_j|$ is odd for $1 \le i < j \le m$. Show that $m \le n$.
(Hint: Consider the cases n odd and n even separately. In the n even case consider the matrix $M = AA^T$ where A is made up from the incidence vectors of the columns of $C_1, C_2, \ldots, C_m$.)
Solution: Let $\mathbf{1}$ denote the all ones vector of dimension n. Let $\mathbf{x}_i, i = 1, 2, \ldots, m$ denote the incidence vectors of the clubs $C_1, C_2, \ldots, C_m$.

   Assume first that n is odd. In which case we let $D_j = [n] \setminus C_j$ for $j = 1, 2, \ldots, n$. Then $|D_j|$ is odd for $j = 1, 2, \ldots, n$ and

$$|D_i \cap D_j| = n - |C_i \cup C_j| = n - |C_i| - |C_j| + |C_i \cap C_j| \text{ is even for } i \neq j.$$

   The result now follows from Odd town.

   The n even case is more tricky. Let A be the $n \times m$ matrix with columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$. Our assumptions imply that (i) $\mathbf{1}^T A = 0$ and so $rank(A) \le n - 1$ and (ii) $B = AA^T = J - I$. We now have the contradiction that $n = rank(B) \le rank(A) \le n - 1$, where $rank(B) = n$ follows from

$$B^2 = J^2 - J - J + I = I.$$

2. Let $A_1, A_2, \ldots, A_m \subseteq [n]$ be such that for $1 \le i < j \le m$, $d_{i,j} = |(A_i \setminus A_j) \cup (A_j \setminus A_i)|$ takes one of two values. Show that $m \le (n+1)(n+4)/2$.
Solution: Let $\mathbf{x}_i$ be the incidence vector of A_i for $i = 1, 2, \ldots, m$. Then $||\mathbf{x}_i - \mathbf{x}_j|| \in \left\{ d_1^{1/2}, d_2^{1/2} \right\}$ where d_1, d_2 are the two values mentioned in the question.

3. Each edge of K_n appears an odd number of times as an edge in the collection $G_1, G_2, \ldots, G_m$ of bipartite subgraphs of K_n. Show that $m \ge (n-1)/2$.
[Hint: Let A_k, B_k, M_k, S be as in the notes on Linear Algebraic Methods and consider the $2n \times n$ matrix $T = \begin{bmatrix} S \\ S^T \\ \mathbf{1}^T \end{bmatrix}$.]
Solution: We deduce from the conditions in the question that

$$S + S^T = J_n - I_n \text{ over } \mathbb{F}_2. \tag{1}$$

   Furthermore, if $m < (n-1)/2$ then $rank(T) \le n - 1$ and there exists $\mathbf{x} \neq 0$ such that $T\mathbf{x} = 0$. But then, (1) implies that $\mathbf{x} = 0$, contradiction.