21-301 Combinatorics
Homework 5
Due: Wednesday, October 14

1. Use the pigeon-hole principle to show that for every integer $k \geq 1$ and prime $p \neq 2, 5$ there exists a power of p that ends with $000 \cdots 0001$ (k 0's).
   Solution: If we consider the infinite sequence $u_\ell = p^\ell \mod 10^{k+1}$ for $\ell = 1, 2, \ldots$, then by the PHP there exist $m < n$ such that $u_m = u_n$, In which case,

   $$p^n - p^m = 10^{k+1}s \text{ or } p^{n-m}(p^m - 1) = 10^{k+1}s$$

   for some positive integer s.

   Now p and 10 are co-prime and therefore $p^m - 1 = 10^{k+1}s'$ for some positive integer s', and this implies the result.

2. Suppose that we two-color the edges of K_6 Red and Blue. Show that there are at least two monochromatic triangles.
   Solution: Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or 6 then there can be at most one Red edge joining x to $1, 2, 3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4, 5$ to $1, 2, 3$. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.

3. Show that $r(C_4, C_4) = 6$ where C_4 denotes a cycle of length four.
   Solution: (a) Color the edges of the 5-cycle (1,2,3,4,5,1) Red and the edges of the remaining 5-cycle (1,3,5,2,4,1) Blue. There are no mono-chromatic 4-cycles.

   (b) If a vertex has Red degree ≤ 2 then its Blue degree is at least 3. So if there are fewer than 3 vertices with Red degree ≥ 3, there are at least 4 vertices of Blue degree ≥ 3.

   (c) Vertex 1 has ≥ 3 Red neighbours X among 3,4,5,6 and vertex 2 has ≥ 3 Red neighbours Y among 3,4,5,6. Now $|X \cap Y| = |X| + |Y| - |X \cup Y| \geq 3 + 3 - 4 = 2$. Suppose then that $3, 4$ are both Red neighbors of 1,2. Then (1,3,2,4,1) is Red.

   (d) Suppose first that at least one of 4,5,6, (4 say), has 2 red neighbors (1,2 say) in 1,2,3. Then (4,1,3,2,4) is Red. Since 1,2,3 each have a red neighbor in 4,5,6, we can assume that the only Red neighbors of 1,2,3 are 4,5,6 in this order. If an edge of (4,5,6) ((4,5) say) is Red then (1,4,5,2,1) is Red. So we can assume that (4,5,6) is Blue. But we know that (3,5) and (3,6) are both Blue and so (3,5,4,6,3) is Blue.