1. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of variables. A literal is a variable $x_i$ or its negation $\bar{x}_i$. A clause $C$ is a set of literals. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a set of clauses, all of size $k$. A variable $x_i$ appears in a clause $C_j$ if $\{x_i, \bar{x}_i\} \cap C_j \neq \emptyset$. An assignment of truth values is a map $\sigma : X \to \{0, 1\}$. We extend $\sigma$ to the set of literals $L$ by putting $\sigma(\bar{x}_i) = 1 - \sigma(x_i)$ for $i = 1, 2, \ldots, n$. We say that $\sigma$ satisfies $\mathcal{C}$ if every clause contains at least one literal $y$ such that $\sigma(y) = 1$. Show that if no variable appears in more than $2^{k-2}/k$ clauses, then there is a least one satisfying assignment.

Solution: Assign a random truth value to each variable. Let $E_i$ be the event that clause $C_i$ is unsatisfied. Thus $P(E_i) = 2^{-k}$. Each event $E_i$ is adjacent to at most $k \times 2^{k-2}/k$ other events. $k$ choices for a variable in $C_i$ times the number of other clauses containing this variable. Thus $d \leq 2^{k-2}$ and so $4dp \leq 1$.

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every $v$ and $c \in S(v)$ there are at most $d$ neighbors $u$ of $v$ such that $c$ lies in $S(u)$. Prove that there is a proper coloring of $G$ assigning to each vertex $v$ a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)

Solution: Assume that each list $S(v)$ is of size exactly $10d$. Randomly color each vertex $v$ with a color $c_v$ from its list $S(v)$. For each edge $e = \{v, w\}$ and color $c \in S(v) \cap S(w)$ we let $E_{e,c}$ be the event that $c_v = c_w = c$. Thus $P(E_{e,c}) = 1/(10d)^2$.

Note that $E_{e,v,w}$ depends only on the colors assigned to $v$ and $w$, and is thus independent of $E_{e,v',w'}$ if $\{v', w'\} \cap \{v, w\} = \emptyset$. Hence $E_{e,v,w}$ only depends on other edges involving $v$ or $w$. Now there are at most $10d^2$ events $E_{e,v,w}$ where $e \in S(v) \cap S(w')$. So the maximum degree in the dependency graph is at most $20d^2$. The result follows from $4 \times 20d^2 \times 1/(10d)^2 < 1$.

3. Suppose that $n \leq 2^{k-3}/k$. Show that $[n]$ can be partitioned into two sets $B, W$ such that neither $B$ nor $W$ contains a $k$-term arithmetic progression i.e. a set $\{a + ib : i = 0, 1, \ldots, k-1\}$.

Solution: Randomly color each $i \in [n]$ with $R$ or $B$. For each $k$-term arithmetic progression $\sigma$ we let $E_\sigma$ be the event that each element of $\sigma$ has the same color. The probability of this is $p = 2^{-(k-1)}$. Two events $E_\sigma, E_\tau$ produce an edge of the dependency graph if $\sigma, \tau$ intersect. Now each $\sigma$ intersects at most $d = k^2 \times n/k = nk$ other progressions. $k^2$ accounts for choosing the $i$th position in $\sigma$ and the $j$th position in the other progression in common. Next observe that there are most $(n-1)/(k-1) \leq n/k$ choices for step-size in the second progression. Thus $4dp \leq 4 \times 2^{k-3} \times 2^{1-k} = 1$. 