21-301 Combinatorics
Homework 3
Due: Wednesday, September 30

1. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of **variables**. A **literal** is a variable x_i or its **negation** $\bar{x}_i$. A **clause** C is a set of literals. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a set of clauses, all of size k. A variable x_i appears in a clause C_j if $\{x_i, \bar{x}_i\} \cap C_j \neq \emptyset$. An **assignment** of truth values is a map $\sigma : X \to \{0, 1\}$. We extend σ to the set of literals L by putting $\sigma(\bar{x}_i) = 1 - \sigma(x_i)$ for $i = 1, 2, \ldots, n$. We say that σ **satisfies** $\mathcal{C}$ if every clause cojntains at least one literal y such that $\sigma(y) = 1$. Show that if no variable appears in more than $2^{k-2}/k$ clauses, then there is a least one satisfying assignment.
   Solution: Assign a random truth value to each variable. Let $\mathcal{E}_i$ be the event that clause C_i is unsatisfied. Thus $P(\mathcal{E}_i) = 2^{-k}$. Each event $\mathcal{E}_i$ is adjacent to at most $k \times 2^{k-2}/k$ other events. k choices for a variable in C_i times the number of other clauses containing this variable. Thus $d \leq 2^{k-2}$ and so $4dp \leq 1$.

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every v and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)
   Solution: Assume that each list $S(v)$ is of size exactly $10d$. Randomly color each vertex v with a color c_v from its list $S(v)$. For each edge $e = \{v, w\}$ and color $c \in S(v) \cap S(w)$ we let $\mathcal{E}_{e,c}$ be the event that $c_v = c_w = c$. Thus $P(\mathcal{E}_{e.c}) = 1/(10d)^2$.

   Note that $\mathcal{E}_{\{v,w\},c}$ depends only on the colors assigned to v and w, and is thus independent of $\mathcal{E}_{\{v',w'\},c'}$ if $\{v', w'\} \cap \{v, w\} = \emptyset$. Hence $\mathcal{E}_{\{v,w\},c}$ only depends on other edges involving v or w. Now there are at most $10d \times d$ events $\mathcal{E}_{\{v,w'\},c'}$ where $c' \in S(v) \cap S(w')$. So the maximum degree in the dependency graph is at most $20d^2$. The result follows from $4 \times 20d^2 \times 1/(10d)^2 < 1$.

3. Suppose that $n \leq 2^{k-3}/k$. Show that $[n]$ can be partitioned into two sets B, W such that neither B nor W contains a k-term arithmetic progression i.e. a set $\{a + ib : i = 0, 1, \ldots, k-1\}$.
   Solution: Randomly color each $i \in [n]$ with R or B. For each k-term arithmetic progression σ we let $\mathcal{E}_\sigma$ be the event that each element of σ has the same color. The probability of this is $p = 2^{-(k-1)}$. Two events $\mathcal{E}_\sigma, \mathcal{E}_\tau$ produce an edge of the dependency graph if σ, τ intersect. Now each σ intersects at most $d = k^2 \times n/k = nk$ other progressions. k^2 accounts for choosing the ith position in σ and the jth position in the other progression in common. Next observe that there are most $(n-1)/(k-1) \leq n/k$ choices for step-size in the second progression. Thus $4dp \leq 4 \times 2^{k-3} \times 2^{1-k} = 1$.