Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2015: Test 2

Name:______________________________

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Q1: (20pts)
Use the Pigeon-Hole Principle to show that if $n$ is odd and $\pi$ is a permutation of $[n]$ then the product $\prod_{i=1}^{n}(i - \pi(i))$ is even.

Solution: Let $n = 2m + 1$. There are $m + 1$ odd numbers in $[n]$ and $m$ even numbers. By the PHP there must be an odd $i$ such that $\pi(i)$ is also odd. But then $i - \pi(i)$ is even and the product itself is even.
Q2: (40pts)
Let \( G = (V, E) \) be a graph of maximum degree \( d \). Let \( V_1, V_2, \ldots, V_r \) be a partition of \( V \) such that \( |V_i| \geq 10d \) for \( i = 1, 2, \ldots, r \). Use the local lemma to show that \( G \) contains a set \( S \) such that (i) \( |S \cap V_i| = 1 \) for \( i = 1, 2, \ldots, r \) and (ii) \( S \) is independent, i.e. contains no edges of \( G \).

**Solution:** We can remove vertices from each \( V_i \) if needed and so we can assume w.l.o.g. that \( |V_i| = 10d \) for \( i = 1, 2, \ldots, r \). Choose \( v_i \) randomly from \( V_i \) for \( i = 1, 2, \ldots, r \) and let \( S = \{v_1, v_2, \ldots, v_r\} \). For an edge \( e = \{x, y\} \in E \) we let \( \mathcal{E}_e \) be the event that both \( x, y \in S \). Thus \( \mathbb{P}(\mathcal{E}_e) \leq p = \frac{1}{100d^2} \). An event \( \mathcal{E}_e \) depends only on events \( \mathcal{E}_f \) for which \( e \) and \( f \) share a common vertex. Thus the dependency graph has degree at most \( 20d^2 \). So, \( 4dp \leq \frac{80d^4}{100d^2} = 1 \).
Q3: (40pts)
(a) Show that in any 3-coloring of the edges of $K_{17}$ there is a monochromatic triangle.

**Solution:** Vertex 1 has degree 16 and so at least one color is used at least 6 times. Let this color be red and suppose that \{1, j\} is red for $j = 2, 3, \ldots, 7$. If there is a red edge \{x, y\} in [2, 7] then \{1, x, y\} is a red triangle. Otherwise [2, 7] is 2-colored and because $R(3, 3) = 6$ it must contain a monochromatic triangle.

(b) Give a 3-coloring of the edges of $K_{10}$ without a monochromatic triangle.

**Solution:** Partition [10] into two 5-sets $S_1, S_2$. Color the edges between $S_1, S_2$ with color 1. This does not create a triangle of color 1. Then because $R(3, 3) = 6$, we can use colors 2, 3 to color the edges in each $V_i$ without creating a monochromatic triangle.