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Probabilistic Method

Coloring problem: $A_1, A_2, \ldots, A_n$ are subsets of $A$. $|A_i| = k$, $i = 1, 2, \ldots, n$.

If $n < 2^{k-1}$ then there exists a partition of $A = R \cup B$ such that

$A_i \cap R \neq \emptyset$ and $A_i \cap B \neq \emptyset$

for $i = 1, 2, \ldots, n$
Note: if \( n \) is too large then it may not be possible to find such a coloring.

Suppose \( n = (\frac{2k}{k}) \) and \( \{A_1, A_2, \ldots, A_n\} = (\frac{\lceil 2k \rceil}{k}) \).

In any "coloring" there is a set \( R \cap B \) of size \( \geq k \). Suppose \( \text{Red} \) then any \( k \)-subset of \( R \) is also red.

The largest possible value of \( n \) (in terms of \( k \)) is not
How can we find a "good" coloring?

We choose $R, B$ uniformly at random and show that with positive probability we get a "good" coloring.

$\text{BAD}(i) = \{ A_i \leq R \lor A_i \leq B \}.$

$\text{BAD} = \bigcup_{i=1}^{n} \text{BAD}(i)$

Want to show that

$P(\text{BAD}) < 1$

$\Rightarrow \exists$ a coloring with desired property.

If there were no coloring then we would have $P(\text{BAD}) = 1$
\[ P(\text{BAD}) = P\left( \bigcup_{i=1}^{n} \text{BAD}(i) \right) \leq \sum_{i=1}^{n} P\left( \text{BAD}(i) \right) \leq \sum_{i=1}^{n} \left( \frac{1}{2} \right)^{k} \leq \frac{n}{2^{k-1}} < 1. \]
Tournaments

A tournament is an orientation of a complete graph $K_n$.

For $1 \leq i < j \leq n$, there is an edge $(i,j)$ or $(j,i)$ but not both.

Let $P_k$ be the property that for every set $S \subseteq [n]$, $|S| = k$, there exists $w \in S$ such that oriented $w$ "beats" $S$ i.e., all edges are from $S \to w$. 
Not every tournament will have property $\mathcal{P}_1$, say.

$$S \leq 3$$ breaks $\mathcal{P}_1$.

But there exist tournaments with $\mathcal{P}_1$.

\[ \text{fill in rest any way} \]

\underline{Theorem}

For every $k$, there exists a tournament with property $\mathcal{P}_k$. 
To prove there exist a tournament with property \( A_k \), we choose \( N \) large and choose our tournament \( T \) at random i.e. we independently orient the edges in either direction.

Bad events: For \( |S| = k \) we let \( B_S = \exists \ w \text{ that beats } S \).

\[
A_{\bar{k}} = \bigcup_{|S| = \bar{k}} B_S
\]

\[
P(\overline{A}_{\bar{k}}) \leq \sum_{|S| = \bar{k}} P(\overline{B}_S)
\]
\[ B_S = \bigcap_{w \in S} B_{S,w} \]

where \( B_{S,w} = \{ w \in S \text{ does not beat } w \} \)

The events \( B_{S,w} \) are independent for a fixed \( S \).

\[ B_{S,w_1} \cap B_{S,w_2} \text{ depend on disjoint "coin flips"} \]

\[ P_r(\neg B_{S,w}) = \left( \frac{1}{2} \right)^k \]
\[ P_r(B_{S,w}) = 1 - \left( \frac{1}{2} \right)^k \]
\[ P_r(\neg B_S) = \left( 1 - \left( \frac{1}{2} \right)^k \right)^{n-k} \]
\[ P_r(T \cap A_{\leq r}) \leq \sum_{s \leq r} P_r(T \cap B_s) \leq \sum_{s \leq r} \left(1 - \frac{1}{2^k}\right)^{n-k} \leq \binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < 1 \text{ if } n \gg k. \]