A₁, A₂, ..., Aₙ are the sets.

We proved that if |Aᵢ| ≥ n and each x ∈ Aᵢ is in \( \leq \frac{n}{2} \) of the Aᵢ then Player 2 can draw via a pairing strategy.

Apply this to \( \times \times \times \times \times \times \times \times \times \times \times \times \) Tic Tac Toe.

1. \( |Aᵢ| = n \) for all i.

2. \# lines through a fixed point \( (c₁, c₂, ..., cₙ) \)
   - n odd: \( \frac{3d - 1}{2} \)
   - Only happens if \( cᵢ = \frac{n+1}{2} \) \( \forall i \)
   - n even: once we fix places where \( cᵢ \) is constant, \( cᵢ \) is not determined
   - \( d = \frac{10}{n+2} \)

\[
\begin{pmatrix}
\cdots & 4 & \cdots & 6 & \cdots & 4 \cdots \\
5 & 5 \\
6 & 4 \\
7 & 3 \\
\end{pmatrix}
\]

\( n = 4 \), \( x = 2 \)

\( n = 16 \), \( x = 4 \), \( y = 6 \)
\textbf{N even:} \# lines \leq \sum_{i=0}^{d-1} (d_i) - 1 = 2^d - 1

\text{places where not constant}

\begin{bmatrix}
  \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}

\begin{align*}
  &c_1, c_2, \ldots, c_d \\
  &3 \ 7 \ \ldots \\
\end{align*}

\[
\frac{n}{2} = \frac{3^d - 1}{2} \quad \text{or} \quad \frac{n}{2} \geq 2^d - 1
\]

\text{n odd} \quad \text{n even}

\[
\begin{bmatrix}
  3 & 1 \\
  3 & \cdot \\
  3 & \cdot \\
  \cdot & \cdot \\
  \cdot & \cdot \\
  3 & n \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 7 & \ldots \\
  2 & 7 & \ldots \\
  3 & 7 & \ldots \\
  4 & 7 & \ldots \\
  \cdot & \cdot & \ldots \\
  \cdot & \cdot & \ldots \\
  \cdot & \cdot & \ldots \\
  N & 7 & \ldots \\
\end{bmatrix}
\]
Odd

\[ f \leq \frac{n+1}{2} \Rightarrow f \leq \frac{n+1}{2} \]

doesn't mean

3
\[
\begin{align*}
\text{If} & \quad \begin{array}{c}
\frac{x}{n} = z \\
\frac{x-1}{n} = y \\
\frac{x-2}{n} = x \\
\end{array} \\
\Rightarrow & \quad n-x+1 = x \\
\Rightarrow & \quad 2 = n-x+1 \\
\Rightarrow & \quad x > x
\end{align*}
\]
So if \( d \) is fixed and we make \( n \) large enough, then player 2 can draw.

If \( n \) is fixed and we make \( d \) large enough then player 1 will win,

Hales-Jewett Theorem.

If we \( r \)-color \( \mathbb{Z}^d \) and \( d \) is sufficiently large then there is a mono-chromatic line. — Ramsey statement.

The Taz Theorem — \( r = 2 \)

Suppose we go on until every point is colored. If \( d \) is large then there must be a line of one color.
Erdős–Selfridge Theorem

If \(|A_i| \geq n\) for \(i \in [N]\) and \(N < 2^{n-1}\) then Player 1 can get a draw.

\[\text{Proof}\]

At any point in the game let \(C_j\) denote the set of elements \(\cap A\) which have been colored.

\(\| = \sum_{1 \leq i \leq 2^n, \forall A_i \neq \emptyset, \forall C_j \neq \emptyset} 2^{-|A_i \cap C_j|}\)

\((\| = \text{expedit number of } \cap \text{ bad sets for } 2)\)

If \(2\) randomly colored.

Suppose that the players choose on

\(x_1, y_1, x_2, y_2, \ldots\)

After first move \(\| \leq N2^{-(n-1)} < 1\)

Now it's player 2's turn.
Claim: Player 2 can keep $\Phi < 1$
for the whole game.

At the end

$$\sum_{i: A_i \cap C_2 = \emptyset} 1 < 1$$

$$\Rightarrow \forall i: A_i \cap C_2 = \emptyset.$$