Tic Tac Toe & Generalizations

The board consists of $[n]^d$

A line is a set of points

$\{x_1^{(1)}, x_2^{(2)}, \ldots, x_d^{(d)}\}$, $d = 1, 2, \ldots, n$

where each $x_j^{(j)}$ is either (i) $k, k, k, \ldots, k$

for some $k \in [n]$, or (ii) $1, 2, \ldots, n$,$\alpha_1^{(1)}$

\[\alpha_{n-1}, \alpha_{n-2}, \ldots, 1\]

$d = 2$

\[
\begin{array}{cccc}
(1, 1) & (1, 1) & (1, 1) & (2, 4) \\
(1, 2) & (2, 1) & (2, 2) & (2, 3) \\
(1, 3) & (3, 1) & (3, 3) & (3, 2) \\
(1, 4) & (4, 1) & (4, 4) & (4, 1)
\end{array}
\]
Lemma 1

The number of winning line is
\[
\frac{(n+2)^d - n^d}{2}
\]

\[X_j^{(1)}, X_j^{(2)}, \ldots, X_j^{(d)} \quad j = 1, 2, \ldots, n\]

n choosing (i)
+ 1 (i1)
+ 1 (i1)

\[
\begin{array}{cccc}
1 & 2 & 4 & 10 \\
1 & 2 & 4 & 10 \\
1 & 2 & 4 & 10 \\
\end{array}
\]

Some point repeated n time.

Two players. Red play (X player) and Blue play, each take lines occupying (coloring) a point. A player wins if he/she colors a complete line.

Draw if nobody colors a line.
لا يمكنني قراءة النص العربي في الصورة.
Lemma 2

Player 1 can always get a draw, at least.

Proof

Strategy Stealing: If player 2 has a winning strategy, then player 1 could make an arbitrary first move, and then win by following player 2's strategy.

A plays \( x \), A imagines playing \( x' \).

Suppose B would win by playing \( y \).

Indeed, A plays \( y' \), and follows B's strategy.
\[ \begin{array}{cccc} z & 8 & 1 & 11 \\ 10 & 0 & 5 & 6 \end{array} \]

\[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \]

\[ P = Z \]

\[ \begin{array}{cccc} 2 \leq r & \leq 10 \\ 2 \leq s & \leq 6 \\ 2 \leq t & \leq 6 \\ 2 \leq u & \leq 6 \end{array} \]

\[ \text{Formulation} \]

\[ \text{Objective : } \min P = z \]

\[ \text{Subject to : } \]

\[ \begin{align*}
  & 2 \leq r & \leq 10 \\
  & 2 \leq s & \leq 6 \\
  & 2 \leq t & \leq 6 \\
  & 2 \leq u & \leq 6
\end{align*} \]
Pairing Strategy

\[
\begin{bmatrix}
11 & 1 & 8 & 1 & 12 \\
6 & 2 & 2 & 9 & 10 \\
3 & 7 & * & 9 & 3 \\
6 & 7 & 4 & 4 & 10 \\
12 & 5 & 8 & 5 & 11 \\
\end{bmatrix}
\]

\(n=5\)

Is there a pairing strategy?

**Lemma 3**

If \(n \geq 2^d - 1\) and \(n\) is odd or \(n \geq 3^{d-1}\) and \(n\) is even then there is a pairing strategy.

\(d=2\)

\(n=3\) or \(n\) odd \(\checkmark\) \(n=5\) \(\checkmark\)

\(n=6\) and \(n\) even \(\checkmark\)

\(n=4\) or 7 \(\times\) are draws


A diagonal vector AB connects the origin to a point on a plane.

Given coordinates of points A and B, we can find the magnitude of vector AB using the distance formula:

\[ \|AB\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

For example, if point A is at (x_1, y_1) and point B is at (x_2, y_2), then the magnitude of vector AB is:

\[ \|AB\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

This formula is useful in various applications, such as in physics for calculating forces and in computer graphics for rendering geometric shapes.
Generalization of Tic-Tac-Toe.

We have a family $F = A_1, A_2, \ldots, A_N \subseteq \mathbb{A}$ ($A_i$ correspond to lines).

A move consists of one player choosing an uncolored cell $a \in \mathbb{A}$ and giving it a color.

A player wins by coloring $r$ whole $A_i$.

\[ \text{In Tic-Tac-Toe : } |A_i| = N \text{ and } N = \frac{(d+2)^n - d^n}{2} \]

A peering strategy is a sequence $x_1, x_2, \ldots, x_{2N}$ of distinct members of $\mathbb{A}$ such that $\{x_{2i-1}, x_{2i}\} \subseteq A_i$ for $1 \leq i \leq N$.

![Diagram of Tic-Tac-Toe with peering strategy and edge labeling]
There is a pairing if there is a matching from RHS into LHS.

Hall's Theorem says that pairing strategy exists iff

$$\bigcup_{A \in C} |A| \geq |G|/2 + \text{GF}$$

Corollary

If $|A_i| \geq n$ and if every $a \in A_i$ is in at most

$$\frac{n}{2}$$

sets of $C$ then $C$ holds.
\[ s_n \leq M \leq \frac{\kappa n}{2} \]