Game $\equiv$ Digraph - acyclic & finite

Partition vertices in $\overrightarrow{N} \cup P$ winning
\[ \uparrow \]
losing

$x \in N$ iff $\overrightarrow{N}(x) \cap P = \emptyset$
\[ \text{set of out neighbors} \]

There is a unique partition that satisfies

Proof is by backwards induction on vertex numbering.
Sums of games

Suppose that we have \( p \) games \( G_1, G_2, \ldots, G_p \) with digraphs \( D_i = (X_i, A_i) \), \( i = 1, 2, \ldots, p \). The sum

\[ G_1 \oplus G_2 \oplus \cdots \oplus G_p \]

is played as follows.

A position is a vector \( (x_1, x_2, \ldots, x_p) \in X \)

\[ X = \times X_1 \times X_2 \times X_3 \times \cdots \times X_p \]

\( x_i \in X_i \)

To make a move a player chooses \( i \) such that \( x_i \) is not a sink of \( D_i \) and then replaces \( x_i \) by

\( y \in N^+(x_i) \).

Choose a pile and remove
Sprague-Grundy numbering.

For \( S \leq \{0,1,2,\ldots,3\} \)

\[
\min(S) = \min \left\{ \frac{3}{2} x \geq 0 : x \notin S \right\}
\]

Given \( D = (X, A) \) with ordering \( x_1, \ldots, x_n \)

we define the Grundy number \( g: X \to \{0,1,2,\ldots\} \)

\( x \in P \iff g(x) = 0 \)

\( x \in P \iff \exists y \in N^+(x) \Rightarrow y \in N \frac{g(y)}{g(y) > 0} \)

Main Theorem

\[
G = G_1 \oplus G_2 \oplus \cdots \oplus G_p
\]

\( G_i \) has Grundy nos. \( g_i \)

\[
g(x_1, x_2, \ldots, x_p) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)
\]

exclusive or addition without carry 1+1 = 0
Take away game  \[ S = 3, 1, 2, 3, 4, 3 \]

\[ g(n) = n \mod 5 \]

**Proof** By induction on \( n \).

\( g(0) = 0 \)

Suppose \( n = 5r + s \) \( \forall 0 \leq s < 5 \)

\[ g(n) = \max \left\{ 0, 1, 2, 3, \frac{s}{3} = 4 \right\} \]

\[ \max \left\{ 0, 1, 2, 4, \frac{s}{3} = 3 \right\} \]

\[ \max \left\{ 0, 1, 3, 4, \frac{s}{3} = 2 \right\} \]

\[ \max \left\{ 0, 2, 3, 4, \frac{s}{3} = 1 \right\} \]

\[ \max \left\{ 1, 2, 3, 4, \frac{s}{3} = 0 \right\} \]
A player can remove any even number of chips
but not the whole pile.
A player can remove the whole pile if it is odd.

\[ g(0) = 0 \]
\[ g(2k) = k - 1 \quad \text{if } k \geq 1 \]
\[ g(2k - 1) = k \]

Assume \( k > 1 \).

\[ g(2k) = \max \{ g(2k-2), g(2k-4), \ldots, g(2), g(0) \} \]
\[ = k - 1 \]

\[ g(2k - 1) = \max \{ g(2k-3), g(2k-5), \ldots, g(3), g(1), g(0) \} \]
\[ = k \]
Ex 1

\[ g(1) = 4 \]

Ex 2

\[ g_2(1) = 5 \]

\[ 4 \oplus 5 = 1 \]

100
\[ \begin{array}{c}
101 \\
001 \\
\end{array} \]

Winning position.

Winning move: Remove 2 from pile 2, \( g_2 \rightarrow 4 \).