\[ \exists N(p,q,r) \text{ such that } y \leq N(p,q,r) \]

and we color \( [n] \) Red/Blue then there exists \( S \subseteq [n] \), \( |S| = p \) and \( (S) \) is Red OR exists \( T \subseteq [n] \), \( |T| = q \) and \( (T) \) is Blue.

We proved this for \( r = 2 \).

Assume inductively that claim is true for \( r - 1 \) and \( r \) with \( p' + q' < p + q \).

Assume \( r \geq 3 \)

\[ n = M + 1, \quad M \text{ very big} \]
M is big enough so that either there is a set \( S \) of size \( p, \) \( p = N \left( p^{-1}, q^{r_1} \right) \), such that \( (\tilde{S}) \) is Red or there is \( T \) of size \( q^r, \) \( q = N \left( p, q^{-1} \right) \), such that \( (\tilde{T}) \) is Blue.

Assume \( S \) exists.

Either there is a Blue \( S' \) of size \( q^r \) or there is a Red \( S'' \) of size \( p^{-1-r} \) and \( (\tilde{S''}) \) is Red.

Then \( S' \cup S'' \in S \) is a Red set of size \( p \).
Applications

Schur's Theorem

\[ \tau_k = N(\underbrace{3, 3, \ldots, 3}_k; 2) = \text{smallest } n \text{ such that if we } k \text{-color the edges of } K_n \text{ then there exists a mono-chromatic } \Delta. \]

Theorem

For all partitions of \([n]\) there exist \(l \) and \(x, y, z \in S_i\) such that \(x+y=z.\)

Proof

\[ n = \tau_k. \] Need to color edges of \(K_n\)

\[ u, v, w \text{ where } |u-v| \leq S_i. \]

\[ x < y < z \]

\[ u = y-x \in S_i; \]

\[ v = z-y \in S_i; \]

\[ W = z-x \in S_i; \]

\[ u + v = w. \]
A set $\mathcal{P}$ points in $\mathbb{R}^2$ is in general position if no 3 points are collinear.

**Theorem**

If $n \geq N(k,k;3)$ and $X$ is a set of $n$ points in $\mathbb{R}^2$ in general position then $X$ contains a $k$-subset that forms a convex polygon.
If every 4 points of a set $Y$ are convex then $Y$ is convex.

We must show that there exists a $k$-set $Y$ such that all 4-subsets of $Y$ are convex.

Coloring: label points of $X = X_1, X_2, \ldots, X_n$ anti-clockwise $X_i$ or clockwise $X_i$

Red

Blue
So there exist \( Y \) such that every \( \triangle \) in \( Y \) has same color.

Claim: \( Y \) defines a convex polygon.

\[
\begin{align*}
\text{Case 1} & \quad a < b < c < d \\
\text{Case 2} & \quad a < b < d < c \\
\text{Case 3} & \quad a < c < b < d
\end{align*}
\]

\[
\begin{align*}
\text{abc is Blue} & \quad \times \\
\text{abd is Red} & \quad \times \\
\text{abc is Blue} & \quad \times \\
\text{abd is Red} & \quad \times \\
\text{acb is Red} & \quad \times \\
\text{acd is Red} & \quad \times \\
\text{bcd is Blue}
\end{align*}
\]
Define $\Gamma(H_1, H_2)$ to minimize $\Gamma$ such that in any 2-coloring of $K_n$, there is a Red $H_1$ or a Blue $H_2$.

We have dealt with case where $H_1, H_2$ are complete.

Claim: $R(P_3, P_3) = 5 \quad [R(\Delta, \Delta) = 6]

R(P_3, P_3) > 4

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array} \]
\( R(P_3, P_3) \leq S. \)