Ramsey's Theorem

Given positive integers $k, l \geq 1$, there exists $R(k, l)$ such that if $n \geq R(k, l)$ then in any two coloring (Red/Blue) of the edges of $K_n$, there exists either a copy of $K_k$ with all Red edges or a copy of $K_l$ with all Blue edges.

Proof

We know that $R(1, k), R(2, k), R(k, 1), R(k, 2)$ exist for all $k$.

We show that

$$R(k, l) \leq R(k, l-1) + R(k-1, l)$$

This shows that $R(k, l)$ exists for all $k, l$, by induction on $k + l$. 

\[ \begin{array}{c}
0, 0 \text{ exists} \\
0, 1 \text{ exists} \\
1, 0 \text{ exists} \\
1, 1 \text{ exists} \\
\end{array} \]
Assume inductively that $R(k-1, l) \neq R(k, l-1)$ exist.

Let $N = R(k-1, l) + R(k, l-1)$.

Take any 2-coloring of $K_n$.

Either (i) $|V_R| \geq R(k-1, l)$ or (ii) $|V_B| \geq R(k, l-1)$

$|V_R| + |V_B| = n - 1$

Assume (i) is true.

Then we a 2-coloring of the edge of $V_R$.

Either $\exists$ Blue $K_k \subseteq V_R$ or $\exists$ Red $K_{k-1} \subseteq V_R$. Add $n$ to get a Red $K_k$.
\[ R(k, l) \leq \binom{k+l-2}{k-1} \]

**Proof**

Induction on \( k+l \).

- **True for** \( k+l = 2 \) as a base case.

1. \( k = 1 \)
   - \( R(1, l) = 1 \)
   - \( l = 1 \)

2. \( k = 2 \)
   - \( R(2, 2) = 2 \)
   - \( l = 2 \)

**Inductive Step:**

\[
R(k, l) \leq R(k-1, l) + R(k, l-1)
\]

\[
\leq \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1}
\]

\[
= \binom{k+l-2}{k-1}
\]

In particular,

\[
R(k, k) \leq \binom{2k-2}{k-1} < 4^k
\]
Lower bound.

\[ R(k, k) \geq 2^{k/2} \]

We show that if \( N \leq 2^{k/2} \) then there is a 2-coloring of \( K_n \) with \( n \) Red \( K_k \) and \( 2^{k/2} \) Blue \( K_k \).

We take a random 2-coloring \( G \) of \( K_n \).

\( C_1, C_2, \ldots, C_N \), \( N = \binom{n}{k} \) are the vertices.

The \( k \)-cliques \( D \) of \( K_n \).

\[ E_{R_{i,j}} = C_j \text{ is Red} \]

\[ E_{R_{i,j}} = C_j \text{ is Blue} \]

\[ E = \bigcup_j E_{R_{i,j}} \]

\[ E = \bigcup_j E_{B_{i,j}} \]
\[ P(E_R \cup E_B) \leq P(E_R) + P(E_B) \]

\[ = 2 P(E_R) \]

\[ = 2 P(\bigcup_{j=0}^{N} E_{R_{ij}}) \]

\[ \leq 2 \sum_{a=1}^{2} P(E_{R_{a,b}}) \]

\[ = 2 \binom{n}{k} \left( \frac{1}{2} \right) \]

\[ \leq 2 \frac{n^k}{k!} \left( \frac{1}{2} \right)^{\frac{k}{2}} \]

\[ \leq 2 \frac{2^{k/2} - (\frac{1}{2})^{\frac{k}{2}}}{k!} \]

\[ = 2 \frac{1 + \frac{k}{2}}{k!} \]

\[ < \frac{1}{1} \]
More general version

Ramsey's theorem is about 2-colorings of
the 2-element subsets of \([n]\)

1. We can use more colors
2. We can color the \(r\)-subset of \([n]\)

**Theorem**

Let \(s \geq 1\), \(q_1, q_2, \ldots, q_s \geq r\) be given

\[
\exists N = \{q_1, q_2, \ldots, q_s; r\} \quad \text{such that} \quad \forall n \geq N
\]

and we color all the \(r\)-subset of \([n]\)

with \(r\) colors then \(\exists 1 \leq j \leq s\) and a set

\(T \subseteq [n]\) such that \(|T| = q_j\) and all

\(r\)-subset of \(T\) are given color \(j\).