21-301 Combinatorics
Homework 9
Due: Friday, November 5

1. Given any sequence of n integers, positive or negative, not necessarily all different, show that some consecutive subsequence has the property that the sum of the members of the subsequence is a multiple of n.

   Solution Let the sequence be $x_1, x_2, \ldots, x_n$ and let $s_i = x_1 + \cdots + x_i \mod n$ for $i = 1, 2, \ldots, n$. If there exists i with $s_i = 0$ then n divides $x_1 + \cdots + x_i$. Otherwise, $s_1, s_2, \ldots, s_n$ all take values in $[n-1]$. By the pigeon-hole principle, there exist $i < j$ such that $s_i = s_j$ and then n divides $x_{i+1} + \cdots + x_j$.

2. Prove that if n is odd then for any permutation π of the set $\{1, 2, ..., n\}$ the product $P(\pi) = (1 - \pi(1))(2 - \pi(2))...(n - \pi(n))$ is necessarily even.

   Solution: A product of integers is even iff at least one of the factors is even. Suppose that $n = 2m + 1$. Let $ODD = \{i : \pi(i) \text{ is odd}\}$. $|ODD| = m + 1$ and there are only m even integers in $[n]$. So there must be an $i \in I$ such that i is odd. But then $\pi(i) - i$ is even.

3. Let G_1, G_2 be fixed graphs. Let $r(G_1, G_2)$ be the smallest integer$N$ such that if we two-color the edges of the complete graph K_N there is a Red copy of G_1 or a Blue copy of G_2, or both. Show that if P_3 is a path of length 3 and C_4 is a 4-cycle then $r(P_3, C_4) = 5$.

   Solution:

   K_4 is the union of an edge disjoint triangle and a copy of $K_{1,3}$ and so $r(P_3, C_4) > 4$.

   Now consider a two coloring of the edges of K_5. Assume that there is no Red copy of P_3.

   Now consider the cycle (1,2,3,4). At most two of its edges can be Red.

   Case 1: (1,2) and (2,3) are both Blue.

   Case 1a: (1,4) and (3,4) are Red.
   This means that (1,5) and (3,5) are both Blue; else we have a Red P_3. So we have a Blue C_4 in (1,2,3,5).

   Case 1b: (3,4) is Blue.
   Consider the edges (2,5),(4,5). If they are both Red then (1,4,5,2) is Red, contradiction. If they are both Blue then (2,3,4,5,2) is Blue.
   Assume next that (4,5) is Blue and (2,5) is Red. If (1,5) is Red then (2,5,1,4) is Red, contradiction. So (1,5) is Blue and (3,5) is Red. If now (1,3) is Red then so is (4,1,3,5), contradiction. So (1,3) is Blue then so is (1,3,4,5,1).
   Now suppose that (4,5) is Red and (2,5) is Blue. If (3,5) is Red then (3,5,4,1) is Red, contradiction. So (3,5) is Blue and (1,5) is Red. If (1,3) is Red then so is (3,1,5,4), contradiction. So (1,3) is Blue and then so is (1,3,5,2,1).

   Case 2: (1,4) and (2,3) are Blue. (We can assume that (1,2) and (3,4) are Red, else we are back in to Case 1).
   (1,3) is Blue, else (2,1,3,4) is Red. Similarly, (2,4) is Blue. But then (1,4,2,3,1) is Blue.