# 21-301 Combinatorics
# Homework 5
# Due: Monday, October 4

1. A bag contains $n$ balls, each of a different color. In a round, a person picks a random ball from the bag, makes a note of its color and then puts it back. What is the expected number of rounds required for the person to have pulled out a ball of each color at least once?

   **Solution:** Let $T_i$ be the number of rounds needed to increase the number of different colors chosen so far from $i-1$ to $i$. Thus $T_1 = 1$ but in general $T_i$ is a random variable and the question asks for $E(T_1 + T_2 + \cdots + T_n)$.

   Now when $i-1$ colors have been chosen, the probability that we see a new one on the next round is $\frac{n-i+1}{n}$, regardless of the previous drawings. Thus $T_i$ is distributed as a geometric random variable with probability of success $\frac{n-i+1}{n}$. Thus

   $$E(T_i) = \frac{n}{n-i+1}$$

   and the expected total number of drawings is

   $$n\sum_{i=1}^{n} \frac{1}{n-i+1} = n\sum_{i=1}^{n} \frac{1}{i}.$$

2. A clown stands at the side of a swimming pool. In his hand is a bag containing $n$ red balls and $n$ blue balls. At each step he puts his hand into the bag and pulls out a random ball and throws it away. If the ball is red, he makes a step towards the pool and if it is blue, he makes a step away from the pool. What is the probability that the clown falls in to the pool?

   **Solution:** Imagine that the pool is to the left of the clown and that the sequence of moves by the clown can be descibed by a sequence $\mathbf{x}$ of $n$ R's and L's. The clown will stay dry iff every prefix $x_1x_2\cdots x_k$ of $\mathbf{x}$ has at least as many R's as L's. The number of choices for $\mathbf{x}$ is then the number of choices of grid paths from $(0,0)$ to $(n,n)$ that never go below the diagonal i.e. $\frac{1}{n+1}\binom{2n}{n}$ and the probability of staying dry is

   $$\frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.$$

3. Let $p = 3k+2$ be prime. Show that every set of positive integers $S$ not containing a multiple of $p$ contains a subset $T$ of size at least $|S|/3$ such that if $x,y,z \in T$ then $x+y \neq z \mod p$.

   (Hint: Let $C = \{k+1, k+2, \ldots, 2k+1\}$ and let $x$ be chosen randomly from $\{1,2,\ldots,p-1\}$. Now consider the number of $s \in S$ such that $xs \mod p$ lies in $C$.)

   **Solution:** We observe first that $C$ is sum free. Let $k+x, k+y, k+z \in C$ where $1 \leq x,y,z \leq k+1$. If $k+x+k+y = k+z \mod p$ then $z = k+x+y \mod p$ which implies that $x$ is at least $k+2$, contradiction.

Next let $S = \{s_1, s_2, \ldots, s_N\}$ and let $Z_i = 1$ if $xs_i \mod p \in C$ and let $Z_i = 0$ otherwise. Then $xs_i$ is equally likely to be any member $\{1, 2, \ldots, p-1\}$. So,

$$\Pr(Z_i = 1) = \frac{k+1}{3k+1} > \frac{1}{3}.$$

So, $E(Z_1 + \cdots + Z_N) > N/3$ and hence ther eexists an $x$ such that if $T = xS \cap C$ then $|T| > N/3$. But then $T$ is sum-free and $x^{-1}T \mod p$ is a subset of $S$ that is sum-free.