21-301 Combinatorics
Homework 3
Due: Monday, September 13

1. Suppose that in the Tower of Hanoi problem there are $n = 2m$ rings and 4 pegs. Show that the rings can be moved from Peg 1 to Peg 4 in at most $3(2^m - 1)$ moves.

   Solution One can proceed as follows: Move the top m rings on peg 1 onto peg 2 in H_m moves, using pegs 1,2,3. Then move the remaining m rings on peg 1 onto peg 4 in H_m moves, using pegs 1,3,4. Then move the m rings on peg 2 onto peg 4 in H_m moves, using pegs 2,3,4. The number of moves used is $3H_m = 3(2^m - 1)$.

2. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form abc satisfies the recurrence $b_0 = 1, b_1 = 3, b_2 = 9$ and

$$\begin{aligned} b_n &= 2b_{n-1} + c_n &(1)\\ c_n &= c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3} &(2)\end{aligned}$$

   where c_n is the number of such sequences that start with a.

   Now find a recurrence only involving b_n, by using (1) to eliminate c_n from (2).

   Solution: There are $2b_{n-1}$ sequences of the required form that start with b or c. There are c_n sequences that start with a. This explains (1).

   There are c_{n-1} sequences that start with aa, b_{n-2} sequences that start with ac, c_{n-2} sequences that start with aba and b_{n-3} sequences that start with abb. This covers the possibilities for sequences starting with a.

   We have

$$b_n - 2b_{n-1} = b_{n-1} - 2b_{n-2} + b_{n-2} + b_{n-2} - 2b_{n-3} + b_{n-3}$$

   and so

$$b_n = 3b_{n-1} - b_{n-3}.$$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

$$a_n + 4a_{n-1} + 3a_{n-2} = n + 1 \quad \text{for } n \geq 2$$

   with initial conditions $a_0 = 1$ and $a_1 = 4$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.

Solution:

$$\begin{aligned} \sum_{n=2}^{\infty}(a_n + 4a_{n-1} + 3a_{n-2})x^n &= \sum_{n=2}^{\infty}(n+1)x^n \\ a(x) - 1 - 4x + 4x(a(x) - 1) + 3x^2 a(x) &= \frac{1}{(1-x)^2} - 1 - 2x \\ a(x)(1 + 4x + 3x^2) &= \frac{1}{(1-x)^2} + 6x \end{aligned}$$

$$
\begin{aligned}
a(x) &= \frac{1}{(1+x)(1+3x)(1-x)^2} + \frac{6x}{(1+x)(1+3x)} \\
&= \frac{23/8}{1+x} + \frac{-69/32}{1+3x} + \frac{5/32}{1-x} + \frac{1/8}{(1-x)^2} \\
&= \sum_{n=0}^{\infty} \left(\frac{23}{8}(-1)^n - \frac{69}{32}(-3)^n + \frac{5}{32} + \frac{1}{8}(n+1) \right) x^n.
\end{aligned}
$$

So

$$
a_n = \frac{23}{8}(-1)^n - \frac{69}{32}(-3)^n + \frac{5}{32} + \frac{1}{8}(n+1) \qquad \text{for } n \geq 0.
$$