9\3\10

Euler's (Totient) function:

$\phi(n)$

of $x \leq n$ such that

$hcf(x, n) = 1$

i.e. x and n are co-prime.

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

for some primes $p_1, \ldots, p_k$

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

Inclusion - Exclusion

x and n are co-prime iff

$p_1 \nmid x$ $\neg$ in A_1

$\nmid \equiv$ not divide

and

$p_2 \nmid x$ $\neg$ in A_2

$\vdots$

and

$p_k \nmid x$ $\neg$ in A_k

$$A_i = \{ x \le n : p_i \mid x \}$$

So

$$\phi(n) = |\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_k|$$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} |A_S|$$

$A_1 = \{x \leq n : p_1 \mid x\}$ $\quad |A_1| = \frac{n}{p_1}$

$A_{\{1,2\}} = \{x \leq n : p_1 \mid x, p_2 \mid x\}$ $\quad |A_{\{1,2\}}| = \frac{n}{p_1 p_2}$

$\vdots$

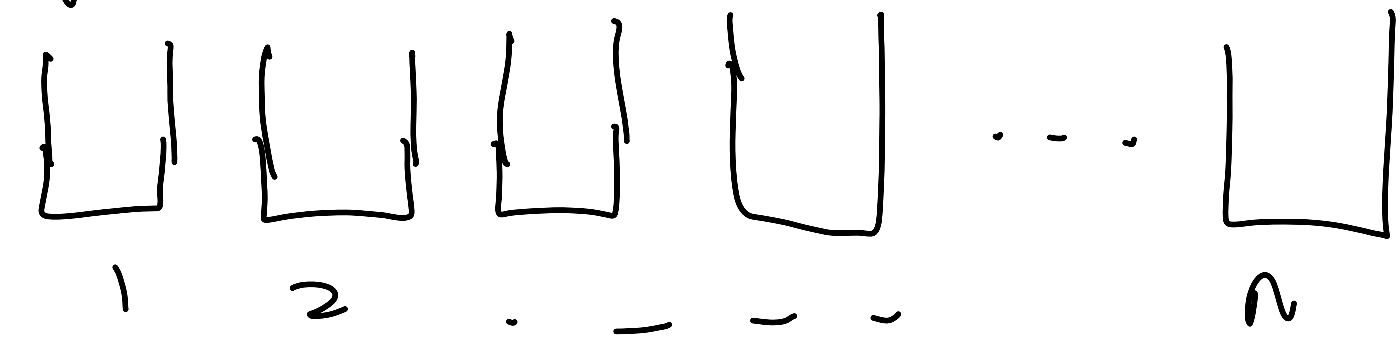
$$|A_S| = \frac{n}{\prod_{i \in S} p_i}$$

$$\phi(n) = \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}$$

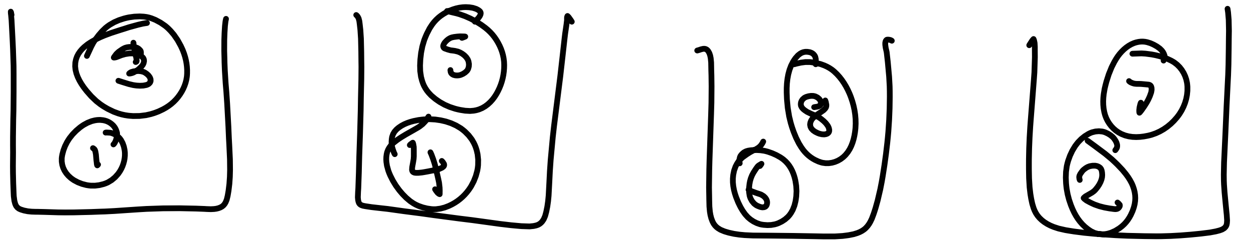
$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

Scrambled Allocations

n boxes



$2n$ distinguishable balls $b_1, b_2, \cdots b_{2n}$ are allocated to the n boxes, 2 to a box.

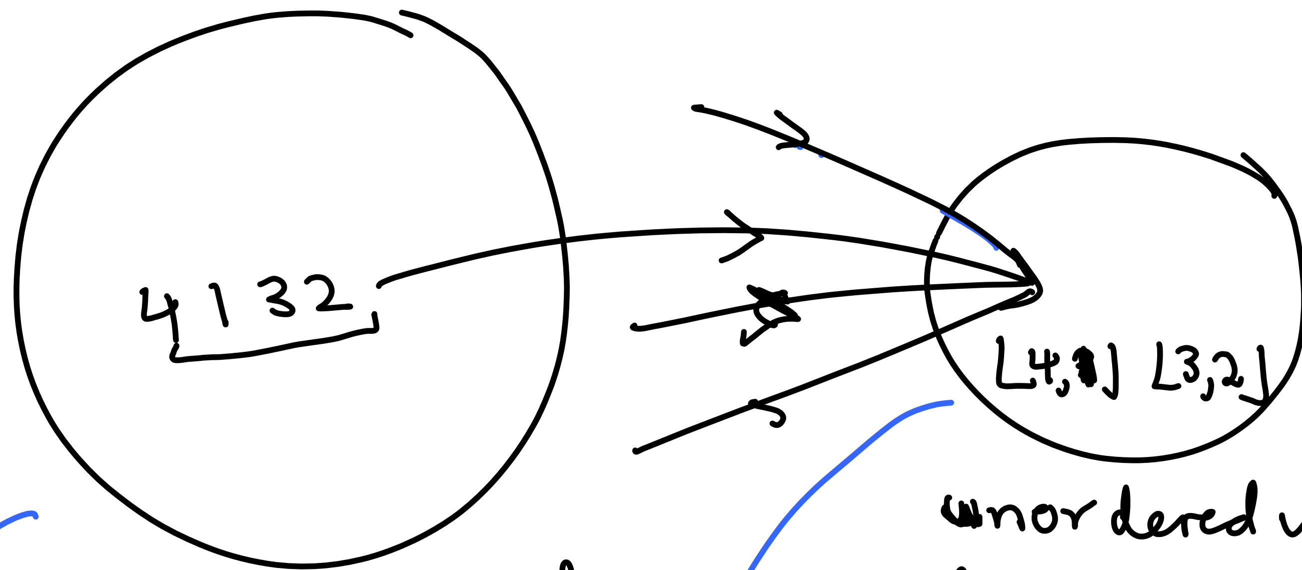


$$\# \text{ allocations} = \frac{(2n)!}{2^n}$$

$\underline{n=8}$ Take a permutation

$$\underbrace{3 \quad 7}_{\text{BOX1}} \quad \underbrace{4 \quad 1}_{\text{BOX2}} \quad \underbrace{8 \quad 2}_{\text{BOX3}} \quad \underbrace{5 \quad 6}_{\text{BOX4}}$$

$n=2$



unordered inside the boxes

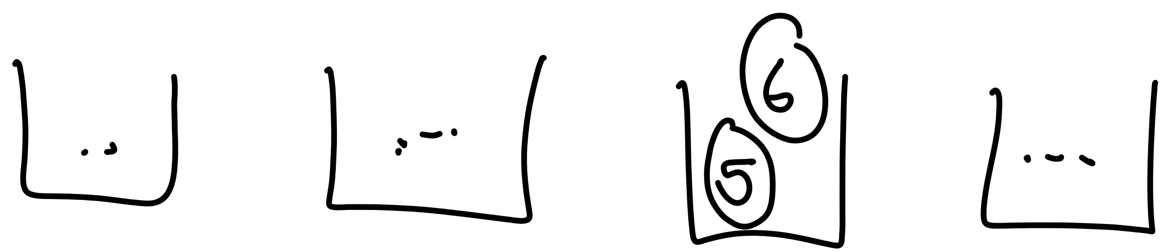
permutations of $[2n]$

size $(2n)!$

Each allocation arises from 2^n permutations

size $\frac{(2n)!}{2^n}$

The allocation is **scrambled** if $\nexists\, i$ such that box i contains b_{2i-1}, b_{2i}



NOT SCRAMBLED

How many scrambled allocations are there?

$\neg$ box 1 contains b_1, b_2

and

$\neg$ box 2 contains b_3, b_4

and

$\vdots$

$A_i = \{$ allocations such box i contains b_{2i-1} and b_{2i} $\}$

Apply the I.E. formula.

$$|A_S| = ?$$

$$A_{\{2,4\}} = \sqcup \; \sqcup_{3,4} \; \sqcup \; \sqcup_{7,8} \; \sqcup \cdots$$

$$|A_{\{2,4\}}| = \frac{(2n-4)!}{2^{n-2}}$$

$$|A_S| = \frac{(2n-2|S|)!}{2^{n-|S|}}$$

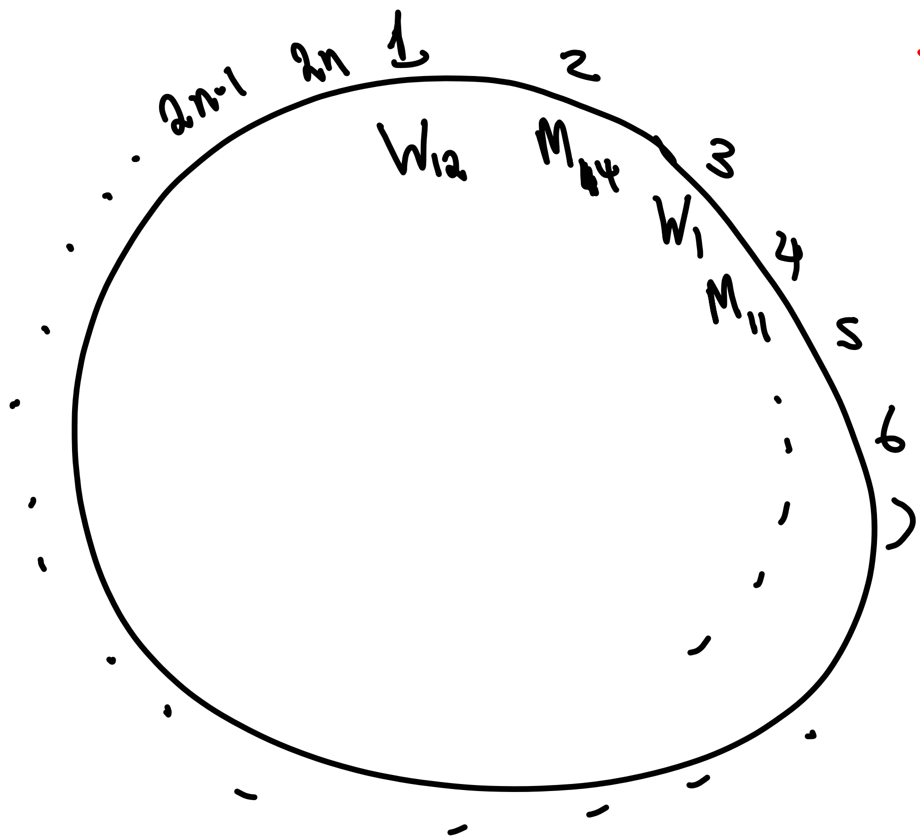
#scrambled =

$$\sum_{S \subseteq [n]} (-1)^{|S|} \frac{(2n-2|S|)!}{2^{n-|S|}}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2n-2k)!}{2^{n-k}}$$

Probléme de Ménages

n couples $M_1, W_1, \ldots, M_n, W_n$



Restriction: M_i & W_i must not sit together.

How many ways?

Avoid

m,
w_1

and

m, w_1

$\neg$ M_1 & W_1 sit together

and

$\neg$ M_2 & W_2 sit together

and

⋮

$A_i = \{$ seating arrangements where M_i and W_i sit together $\}$

Suppose $|S| = k$

placement of rest

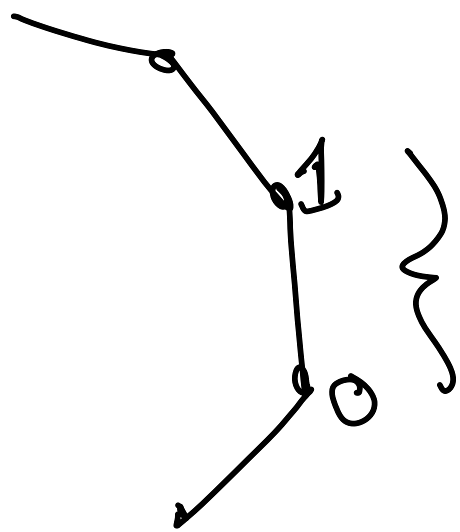
$$|A_S| = 2 \times k! \times (n-k)!^2 \times d_k$$

Whether or not men are in the odd seats

how the k pairs are ordered round the table

ways of putting the k couples together (in ...

d_k = # of ways of putting k 1's onto a $2n$-cycle such that each 1 is separated by at least one 0. $= \frac{2n}{k}\binom{2n-k-1}{k-1}$



couple go here

The order $\begin{matrix}M\\W\end{matrix}$ or $\begin{matrix}W\\M\end{matrix}$ is determined by which sex goes into odd seat