9\1\10

$A_1, A_2, \ldots, A_N \subseteq A$

$$\left| \bigcap_{i=1}^{N} \overline{A_i} \right| =$$

$$|A| - |A_1| - |A_2| - \ldots$$
$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + \ldots$$
$$- |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| \ldots$$

Notation:

$S \subseteq [N]$ (S is a set of indices, NOT a subset of A)

$$A_S = \bigcap_{i \in S} A_i$$

$$A_{\{1,3,9\}} = A_1 \cap A_3 \cap A_9$$

$$A_{\{7,11,59,63\}} = A_7 \cap A_{11} \cap A_{59} \cap A_{63}$$

$$A_\emptyset = A$$

$$\left|\bigcap_{i=1}^{N}\overline{A}_i\right| = \sum_{S\subseteq[N]}(-1)^{|S|}\left|A_S\right|$$

Derangements

A derangement of $[n]$ is permutation π such that

$$\pi(i) \neq i \quad , \; i = 1, 2, \dots, n.$$

D_n = set of derangements of $[n]$

D_n = not something 1   not $\pi(1) = 1$

and

not something 2   not $\pi(2) = 2$

and

⋮

$$A_i = \{\pi : \pi(i) = i\}$$

$$D_n = \bigcap_{i=1}^{n} \overline{A_i}$$

$$|D_n| = \sum_{S \subseteq [n]} (-1)^{|S|} \, |A_S|$$

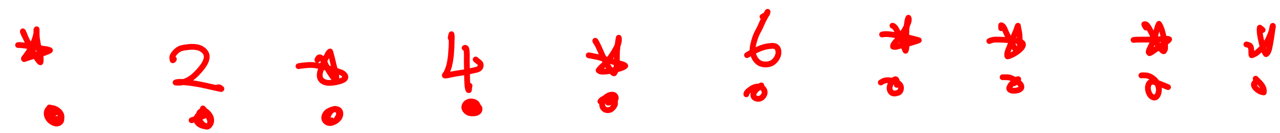
$$|A_\emptyset| = n!$$

$$|A_{\{1\}}| = \left| \{ \pi : \pi(1) = \underline{1} \} \right|$$

$$\underline{1} \quad * \quad * \quad * \quad * \quad * \quad * \quad *$$

$$|A_{\{1\}}| = (n-1)!$$

$$|A_{\{2,4,6\}}|$$



$$|A_{\{2,4,6\}}| = (n-3)!$$

$$|A_S| = (n-|S|)!$$

$$|D_n| = \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)!$$

$$= \sum_{k=0}^{n} \sum_{|S|=k} (-1)^{|S|} (n-|S|)!$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!$$

sets, $|S| = k$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!$$

sets, $|S| = k$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= n! \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{n!}\right) \approx \frac{n!}{e}$$

Proof of Inclusion-Exclusion Formula

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$$(1-\theta_{x,1})(1-\theta_{x,2})\cdots\cdots(1-\theta_{x,N}) = \begin{cases} 1 & x \in \bigcap \bar{A}_i \\ 0 & \text{otherwise} \end{cases}$$

$$\left|\bigcap_{i=1}^{N} A_i\right| = \sum_{x \in \bar{A}} (1-\theta_{x,1})(1-\theta_{x,2})\cdots\cdots(1-\theta_{x,N})$$

$$= \sum_{x \in A} \sum_{S} (-1)^{|S|} \prod_{i \in S} \theta_{x,i}$$

where I choose θ term

$$= \sum_{x\in A}\sum_{S}(-1)^{|S|}\prod_{i\in S}\theta_{x,i}$$

$$= \sum_{S\subseteq[N]}(-1)^{|S|}\sum_{x\in A}\underbrace{\prod_{i\in S}\theta_{x,i}}_{1 \text{ iff } x\in A_S}$$

$$= \sum_{S\subseteq[N]}(-1)^{|S|}\,|A_S|.$$

Surjections

Fix n, m.

$A = \{ f : [n] \longrightarrow [m] \}$

$|A| = m^n$

$F(n,m) = \{ f \in A : f \text{ is onto } [m] \}$

$\neg \equiv$ not

$$F(n,m) = \{ f :$$

$$\neg \boxed{\nexists i : f(i) = 1} \leftarrow A_1$$

and $$\neg \boxed{\nexists i : f(i) = 2} \leftarrow A_2$$

and $$\neg \boxed{\nexists i : f(i) = 3} \leftarrow A_3$$

$$\vdots$$

$$A_i = \{ f : \nexists i : f(i) = 1 \}$$

$= \{ f$ that miss $i \}$

$$|F(n,m)| = \sum_{S \subseteq [m]} (-1)^{|S|} \underbrace{|A_S|}_{??}$$

$$A_S = \{ f : \text{miss all of } S \}$$

$$= \{ f : [n] \to [m] \setminus S \}$$

$$|A_S| = (m - |S|)^n$$

$$|F(n,m)| = \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$