9\16\10

Events

$A \subseteq \Omega$ is called an event.

$$P(A) = \sum_{\omega \in A} P(\omega).$$

Pennsylvania Lottery

You choose 7 numbers I.

State randomly chooses 11 numbers J from $[80]$

$$\mathrm{WIN} = \{J : J \supseteq I\}$$

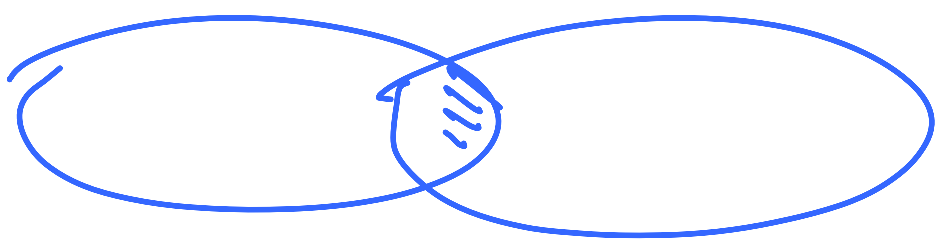
$$= \frac{\#\{J : J \supseteq I\}}{\# J} = \frac{\binom{73}{4}}{\binom{80}{11}}$$

Boole's Inequality

$$P(A \cup B) \leq P(A) + P(B)$$

2 events A, B

$$= P(A) + P(B) - P(A \cap B)$$



$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Induction True for $k=2$

General k: $A = \bigcup_{i=1}^{k-1} A_i$ & $B = A_k$

$$P\left(\bigcup_{i=1}^{k} A_i\right) = P(A \cup B) \leq P(A) + P(B)$$

induction

$$\leq \sum_{i=1}^{k-1} P(A_i) \quad + P(A_k)$$

Coloring Problem

"Probabilistic Method"

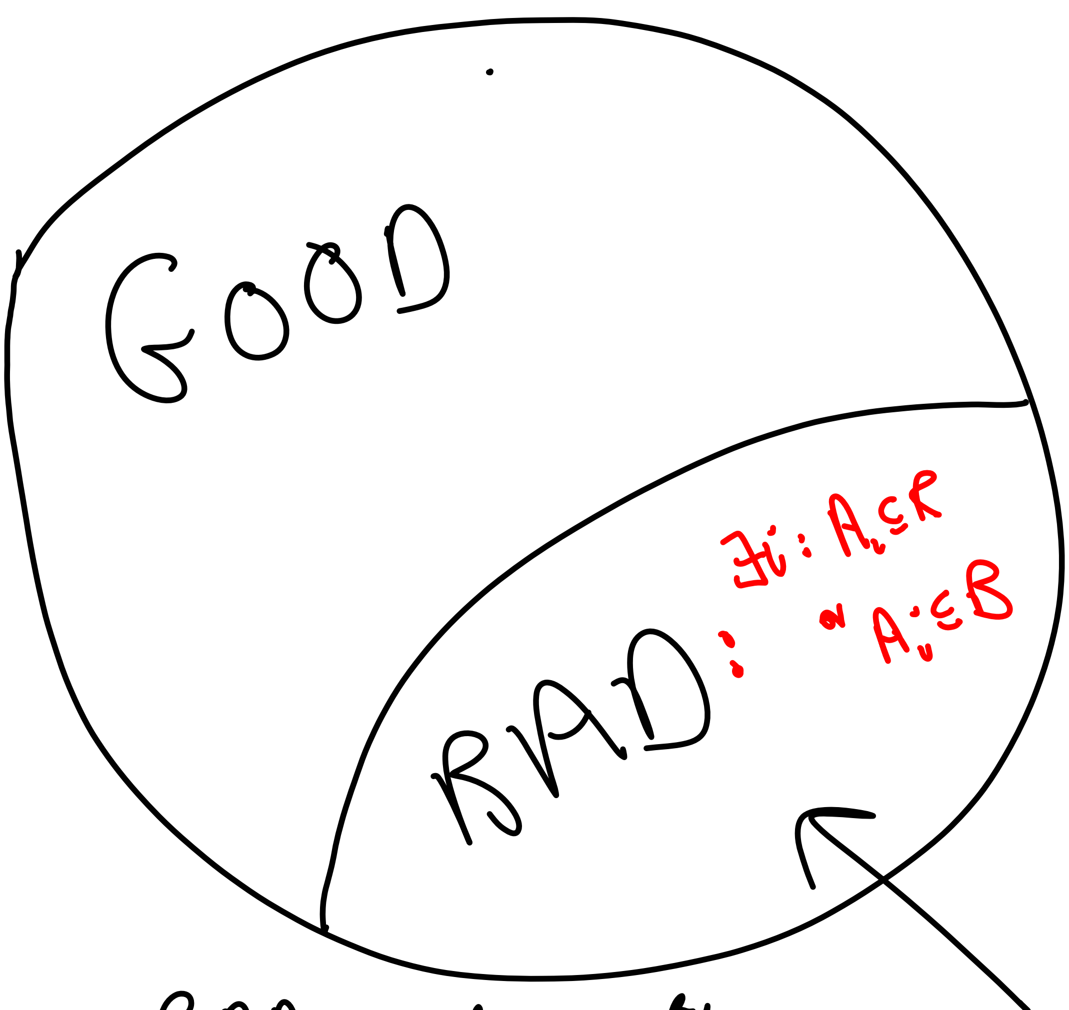
$A_1, A_2, \ldots, A_n \subseteq A$

and $|A_i| = k$ for $i = 1, 2, \ldots, n.$

$$n < 2^{k-1}$$

Claim: $\exists$ partition of A into $R \cup B$

s.t. $A_i \cap R \neq \emptyset$ & $A_i \cap B \neq \emptyset$ $\forall i$



Claim: BAD is not everything

$\equiv$ $P(BAD) < 1$

if I chose a random coloring

Estimate $P(BAD)$ via Boole's inequality.

$$BAD = \bigcup_{i=1}^{n} \underbrace{BAD(i)}_{A_i \subseteq R \text{ or } A_i \subseteq B}$$

$$P(BAD) \leq \sum_{i=1}^{n} P[BAD(i)]$$

$$= \sum_{i=1}^{n} \frac{1}{2^{k_0-1}}$$

$$= \frac{n}{2^{k_0-1}} < 1 \quad \text{by assumption.}$$

If n is too large then it is not possible to 2-color the sets.

$n = \binom{2k}{k}$ and $A = [2k]$

and $A_1, A_2, \ldots, A_n = \binom{[2k]}{k}$

Take any partition of A into $R \cup B$

We can assume $|R| \geq k$ (or $|B| \geq k$)

Choose any k-subset of R

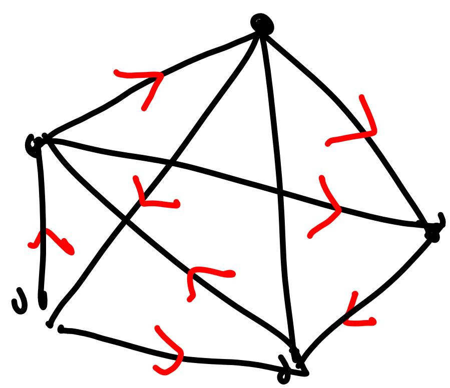
This is A_i for some i — ie coloring is BAD

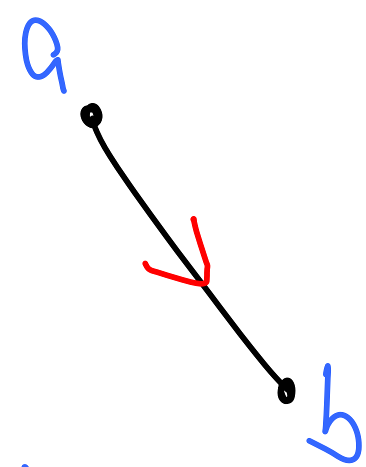
Tournaments

n players.

Each player plays every other player.

n = 5





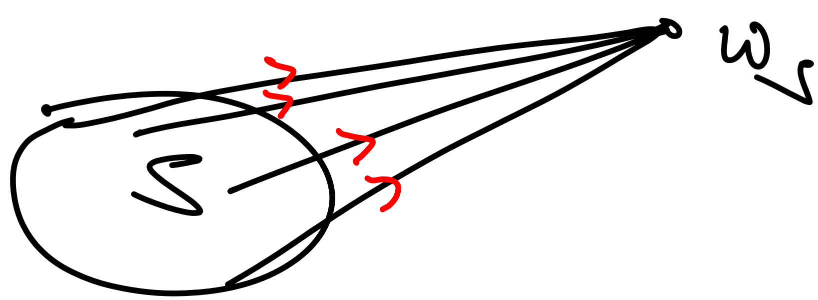
b beats a

Fix k

Property B_k: $\forall S$, $|S| = k$,

$\exists$ a person w_S who beats

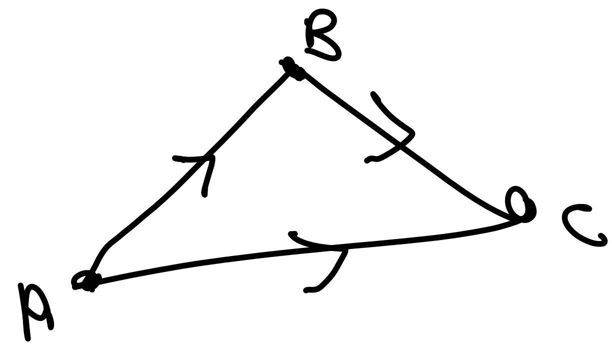
everyone in S.

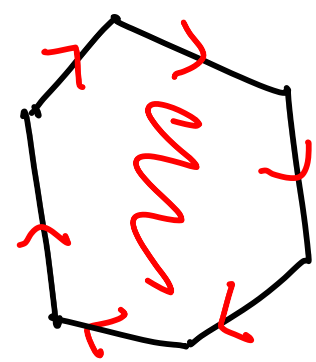


Do there exist tournaments with property B_k.

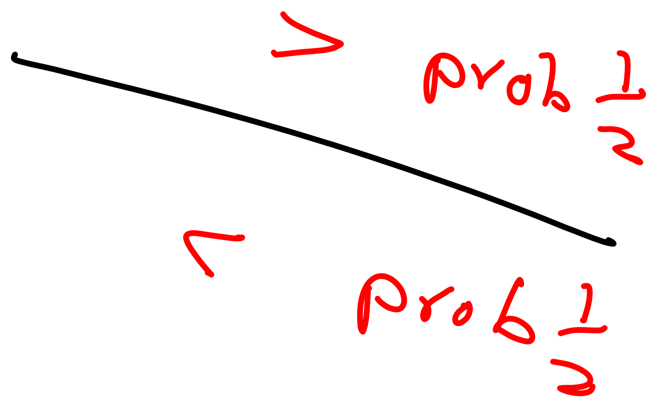
Not every tournament has property B_k

$k=1$





To show $\exists$ tournaments with property B_k: we pick a large n and choose a tournament at random:



each edge.

We use Boole's inequality to estimate probability that tournament does not have B_k.

$\mathcal{E}$ = not having B_k

$$\mathcal{E} = \bigcup_{|S|=k} \mathcal{E}_S$$

$\nexists\, w_S$

$$\Pr(\mathcal{E}) \le \sum_{|S|=k} \Pr(\mathcal{E}_S)$$

$$= \binom{n}{k} \Pr(\mathcal{E}_S)$$

$\Pr(\odot \text{ beats } S) = \frac{1}{2^k}$

$$\Pr(\mathcal{E}_S) = \left(1 - \frac{1}{2^k}\right)^{n-k}$$

$$\Pr(\mathcal{E}) \le \binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k}$$

$$\longrightarrow 0$$

as $n \to \infty$, if k is fixed.