9\10\10

Towers of Hanoi

$$H_n = 2H_{n-1} + 1$$

$H_1 = 1$

$H_2 = 3$

$H_3 = 7$

$\vdots$

$$H_n = 2^n - 1$$

Proof: generating functions

Induction:

$$H_{n+1} = 2(2^n - 1) + 1 \qquad \text{induction.}$$
$$= 2^{n+1} - 1$$

Inhomogeneous problem

$$a_n - 5a_{n-1} + 6a_{n-2} = n^2, \quad n \geq 2$$

$$a_0 = a_1 = 1$$

$$\underbrace{\sum_{n=2}^{\infty} a_n x^n}_{a(x)-1-x} - 5\underbrace{\sum_{n=2}^{\infty} a_{n-1} x^n}_{x(a(x)-1)} + 6\underbrace{\sum_{n=2}^{\infty} a_{n-2} x^n}_{x^2 a(x)} = \sum_{n=2}^{\infty} n^2 x^n$$

$$\sum_{n=2}^{\infty} n^2 x^n = \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=2}^{\infty} n x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

differentiate

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n$$

differentiate

$$\frac{2}{(1-x)^3} = 2 + 3 \times 2 x + \cdots + (n+2)(n+1)x^n + \cdots$$

$$\frac{1}{(1-x)^3} = 1 + \frac{3 \times 2}{2} x^2 + \cdots + \frac{(n+2)(n+1)}{2} x^n + \cdots$$

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$= x^2\left(2 + 3\times 2 \times x + \cdots\right.$$

$$= \frac{2x^2}{(1-x)^3}$$

$$\sum_{n=2}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

Another way

$$\sum_{n=2}^{\infty} n^3 x^n$$

$$= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{(1-x)^4}$$

Use linear equations to figure out A, B, C, D

Back to recurrence,

$$[a(x) - 1 - x] - 5x[a(x) - 1] + 6x^2 a(x)$$

$$= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

Tidy up:

$$a(x)\underbrace{\left(1 - 5x + 6x^2\right)}_{(1-2x)(1-3x)} = 1 + x - 5x + \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$a(x) = \frac{1-4x}{(1-2x)(1-3x)} + \frac{2x^2}{(1-x)^3(1-2x)(1-3x)} + \frac{x}{(1-x)^2(1-2x)(1-3x)}$$

$$a(x) = \frac{1-4x}{(1-2x)(1-3x)} + \frac{2x^2}{(1-x)^2(1-2x)(1-3x)} + \frac{x}{(1-x)^3(1-2x)(1-3x)}$$

$$= \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{(1-x)^3} + \frac{D}{(1-x)^2} + \frac{E}{1-x}$$

Figure out A, B, C, D, E

Newton's Binomial Theorem

$$a(x) = (1+x)^{\alpha}$$

α is any real or complex number

$$= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

Taylor's Series

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$a(x)\,b(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Derangements

d_n = # of derangements of $[n]$.

$$n! = \underbrace{\sum_{k=0}^{n} \binom{n}{k} d_{n-k}}$$

# of permutations π with k "fixed points"

Fixed point: $\pi(i) = i$

i	1	2	$\boxed{3}$	4	5	6	$\boxed{7}$	8
$\pi(i)$	2	4	$\boxed{3}$	5	6	8	$\boxed{7}$	1

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}$$

$$\underbrace{1}_{c_n} = \sum_{k=0}^{n} \underbrace{\frac{1}{k!}}_{a_k} \cdot \underbrace{\frac{d_{n-k}}{(n-k)!}}_{b_{n-k}}$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$\underbrace{1}_{c_n} = \sum_{k=0}^{n} \underbrace{\frac{1}{k!}}_{a_k} \cdot \underbrace{\frac{d_{n-k}}{(n-k)!}}_{b_{n-k}}$$

Multiply by x^n and add

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n$$

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n$$

$$\begin{tikzcd}[column sep=small]
\sum_{n=0}^{\infty} x^n & & & \\
c(x) \arrow[u] \arrow[dd] & = & a(x) \arrow[d] & b(x) \arrow[d] \\
& & \sum_{n=0}^{\infty} \frac{x^n}{n!} & \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n \arrow[d] \\
\frac{1}{1-x} & = & e^x \times & d(x)
\end{tikzcd}$$

$$d(x) = e^{-x} \cdot \frac{1}{1-x}$$

coefficient of x^n

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \cdot 1$$

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$