8\30\10

Multi- Sets

BANANA

How many ways are there of re-arranging the letters.

$\{$letters in BANANA$\} = \{A, B, N\}$

multi-set $\{3 \times A, 1 \times B, 2 \times N\}$

Counting permutations of BANANA

$B_1 A_1 N_1 A_2 N_2 A_3 \Rightarrow$ BANANA

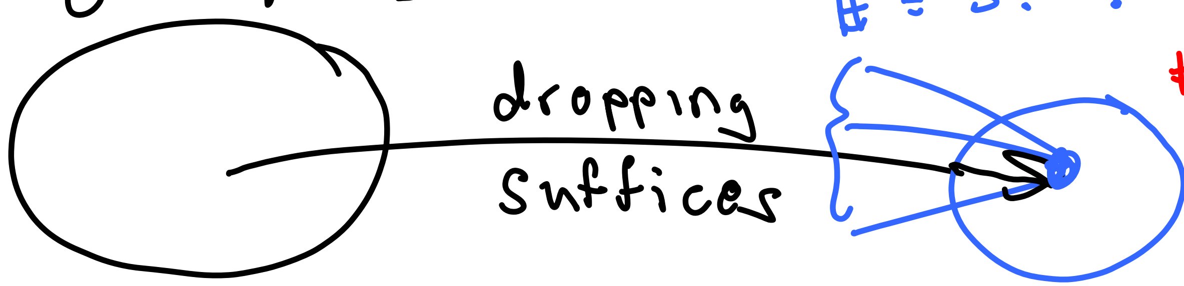
$\#$ permutations of $\uparrow$ $= 6!$

$B_1 A_1 N_1 A_3 N_2 A_2 \Longrightarrow$ BANANA

$A_3 A_2 A_1 N_1 N_2 B \Longrightarrow$ AAANNB

6! sequences

$\# = 3!\,1!\,2!$



$\#$ sequence $= \dfrac{6!}{3!\,1!\,2!}$

General Case:

$$\{a_1 \times 1, a_2 \times 2, \ldots, a_n \times n\}$$

$$a_1 + a_2 + \cdots + a_n = m$$

$$\#\text{permutations} = \frac{m!}{a_1!\, a_2! \cdots a_n!}$$

Multinomial Coefficients

$$\binom{m}{a_1, a_2, \ldots, a_n} = \frac{m!}{a_1! \cdot a_2! \cdots a_n!}$$

$a_1 + a_2 + \cdots + a_n$

$$\binom{n}{k} = \binom{n}{k, n-k}$$

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{\substack{a_1 + \cdots + a_n = m \\ a_1, \cdots a_n \geq 0}} \binom{m}{a_1, \cdots a_n} x_1^{a_1} \cdots x_n^{a_n}$$

Why?

$$(x_1 + x_2 + x_3)^6 =$$

$$(x_1 + x_2 + x_3)(x_1 + x_2 + x_3) \cdots (x_1 + x_2 + x_3)$$

$$x_1 x_1 x_1 x_1 x_1 x_1 + x_1 x_1 x_1 x_1 x_2 + x_1 x_1 x_1 x_1 x_1 x_3 +$$

$$+ x_1 x_1 x_1 x_2 x_1 + \cdots + x_3 x_3 x_3 x_3 x_3 x_3$$

$\vdots$

$x_2 x_2 x_3 x_1 x_2 x_3$

$\vdots$

$x_3 x_2 x_2 x_1 x_3 x_2$

$\vdots$

$? \, x_1 x_2^3 x_3^2$

$$= \frac{6!}{1!\,3!\,2!} = \binom{6}{1,3,2}$$

# of permutations of $\{1 \times 1, 3 \times 2, 2 \times 3\}$

Putting $x_1 = \cdots = x_n = 1$

$$n^m = \sum_{\substack{a_1 + \cdots + a_n = m \\ a_1, \ldots a_n \geq 0}} \binom{m}{a_1, \ldots, a_n}$$

Balls into boxes

m distinguishable balls.

n boxes.

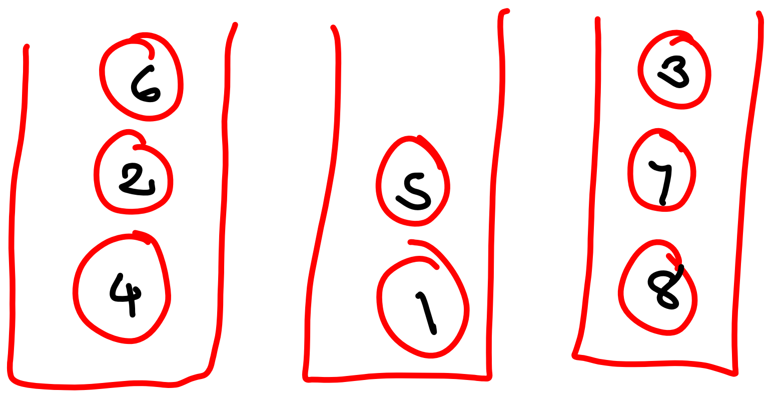
How many ways are there of putting m balls into n boxes so that box i get b_i balls?

Answer: $\binom{m}{b_1, \dots, b_n}$ why?

Establish relationship

Allocations $\longleftrightarrow$ Permutations

of $b_1 \times 1, b_2 \times 2, \dots b_n \times n$



2 1 3 1 2 1 3 3

balls

Inclusion-Exclusion Formula:

2 sets $\quad A_1, A_2 \subseteq A$

$|\overline{A}_1 \cap \overline{A}_2| =$ # of elements not in either A_1 or A_2

$$\parallel$$

$$|A| - |A_1 \cup A_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3-seb

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| =$$

$$|A|$$

$$- |A_1| - |A_2| - |A_3|$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$

$$- |A_1 \cap A_2 \cap A_3|$$

Simple Example

How many integers in $[10{,}000]$ are not divisible by 4, 6 or 10

$A_1 = \{x : 4 \mid x\}$ $A_3 = \{x : 10 \mid x\}$

$A_2 = \{x : 6 \mid x\}$

We want

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$$

$$= 10000$$

$$-2500 - 1666 - 1000 \qquad (-|A_1| - |A_2| - |A_3|)$$

$$+ 833 + 50 + 333 \qquad (+|A_1 \cap A_2| + \cdots)$$

$$- 166 \qquad (-|A_1 \cap A_2 \cap A_3|)$$

$$=$$