8/27/10

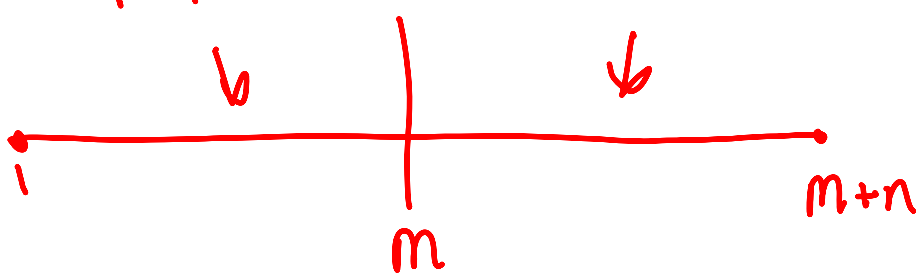
Vandermonde's Identity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}$$

r from ↓

k-r from ↓

# of k-sets in [m+n]



$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

$(1+x)(1+x)(1+x) \cdots\cdots (1+x)$

Coefficient of x^r =

# of ways of chose x from $(1+x)$, r times.

$$(1+a)(1+a)(1+a) =$$

$$1\times1\times1 + 1\times1\times a + 1\times a\times1 + a\times1\times1 + \cdots$$

Substitute values

$\underline{x=1}$

$$\underbrace{(1+1)^n}_{2^n} = \sum_{r=0}^{n} \binom{n}{r}$$

\# of subsets @ [n]

$\underline{x=1}$

$$(1-1)^n = 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots$$

= # even sets − # odd sets

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

differentiate

$$n(1+x)^{n-1} = \sum_{r=0}^{n} r \binom{n}{r} x^{r-1}$$

<u>$x = -1$</u>

$$0 = -\binom{n}{1} + 2\binom{n}{2} - 3\binom{n}{3} + \cdots$$

$$\binom{n}{1} + 3\binom{n}{3} + \cdots = 2\binom{n}{2} + 4\binom{n}{4} + \cdots$$

(i) $\mathrm{PATHS}(a,b) =$

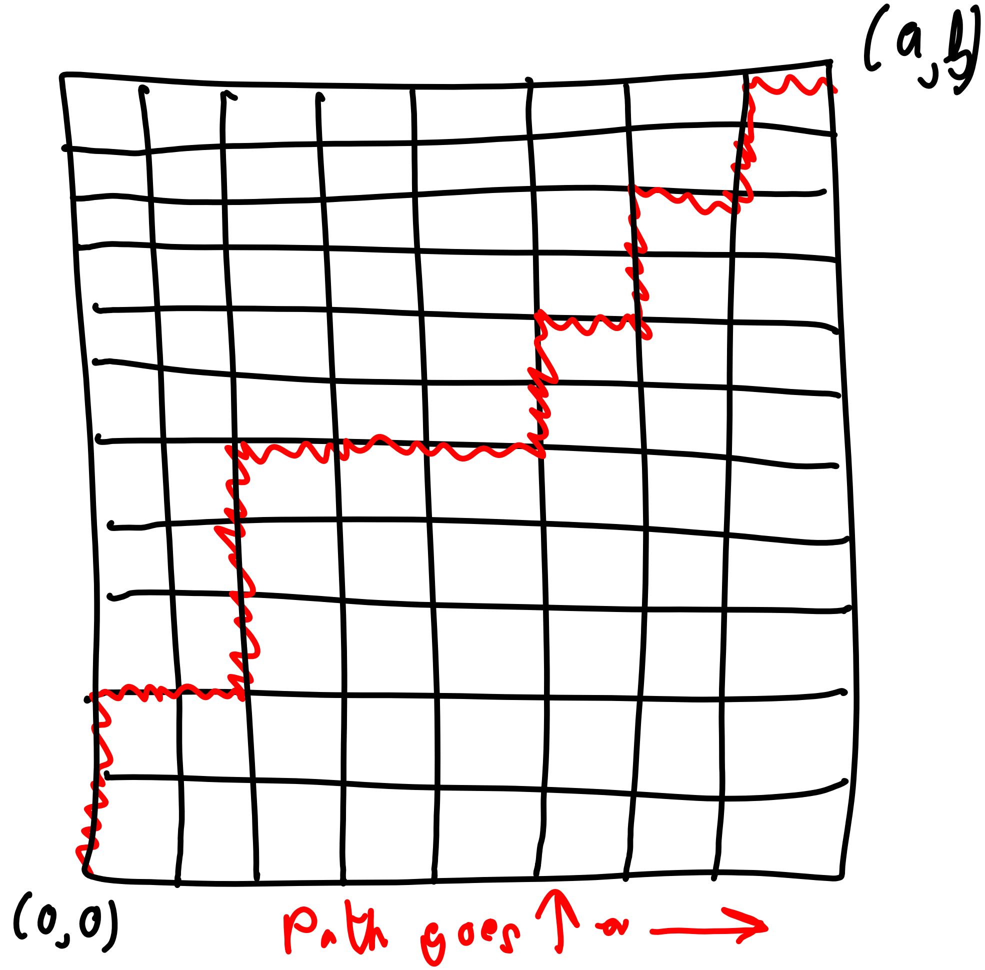
$\{\text{paths } (0,0) \to (a,b)\}$

(ii) $a < b$

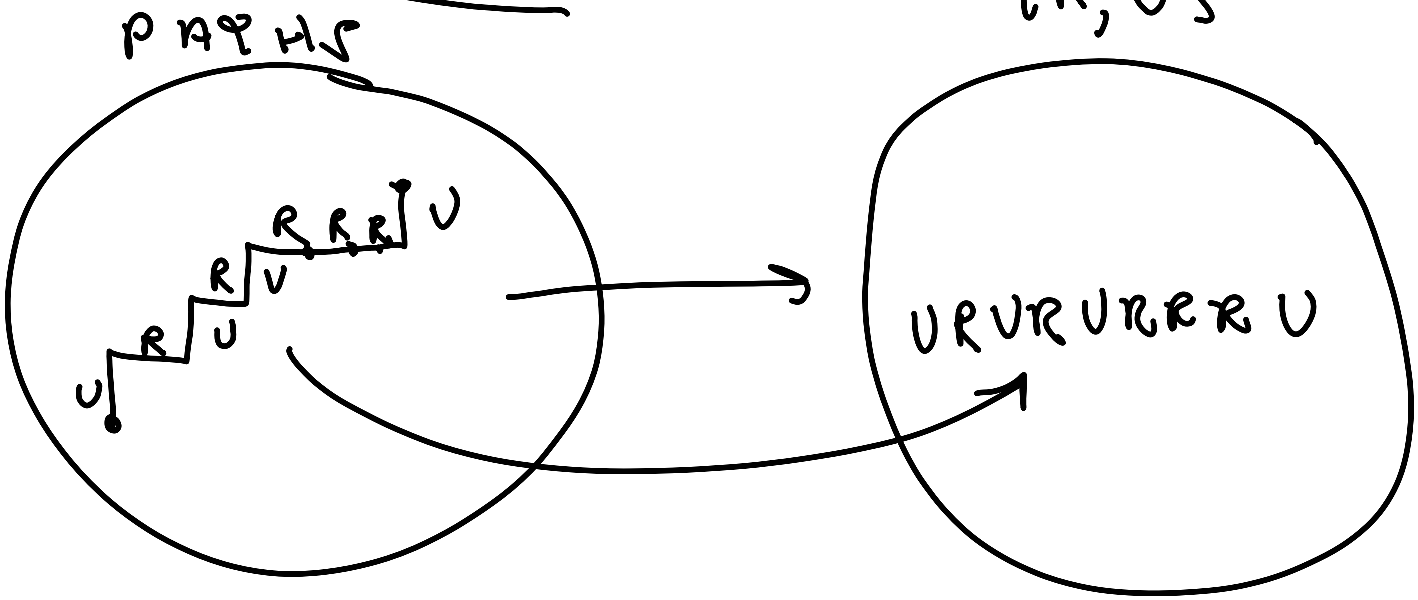
$\mathrm{PATHS}_{>}(a,b) =$

$\{$paths that stay strictly above the diagonal $(1,1), (2,2), \ldots$

(iii) $\mathrm{PATH}_{\geq}(a,b)$

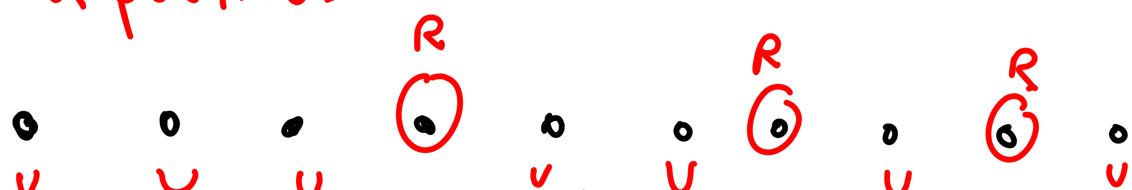


(1) $\underline{\text{PATHS}(a,b)}$



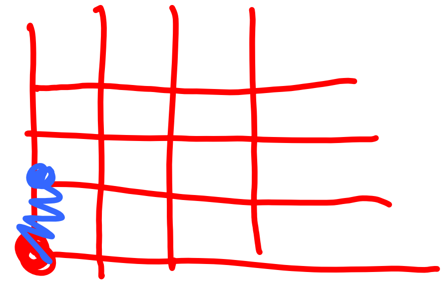
#path = #strings of $a * R$ &
$b * U$

choose a position



$a+b$ is

(2) $\text{PATHS}_{>}(a)$



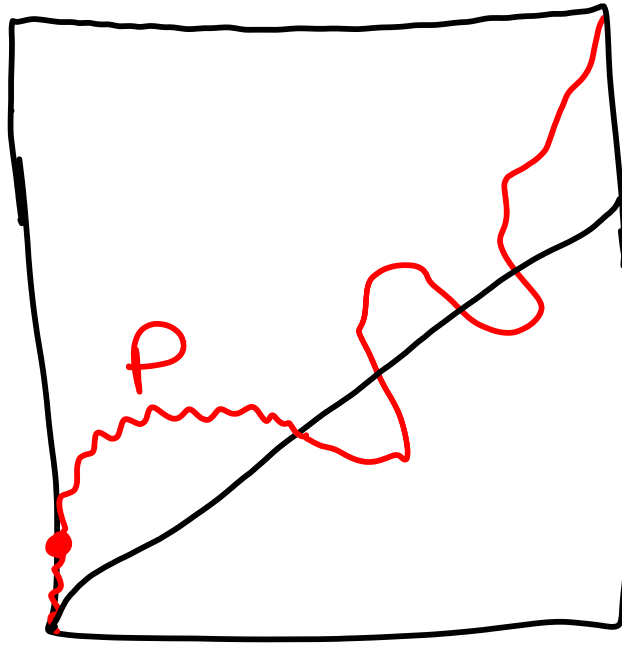
$|\text{PATHS}_{>}(a,b)| =$

$|\text{PATHS}((0,1) \to (a,b))|$

$- |\text{PATHS}((0,1) \to (a,b)$ that cross diagonal $|$

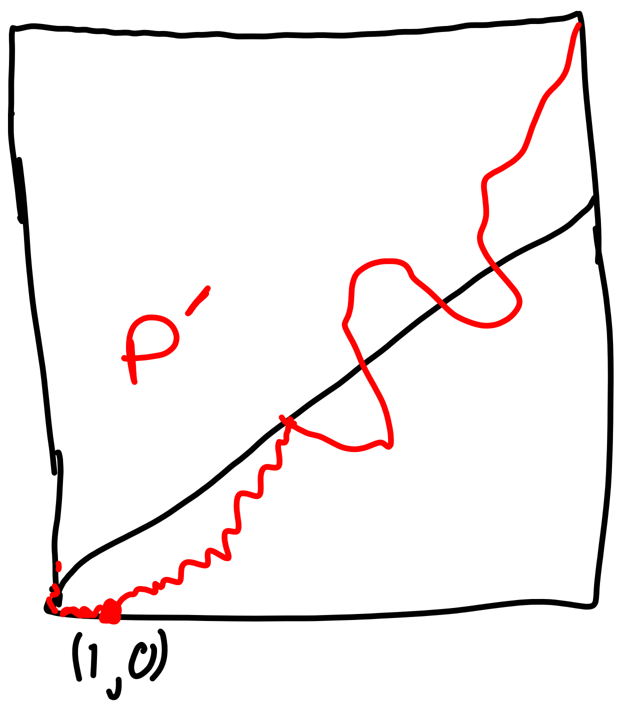
$|\text{PATHS}((0,1) \to (a,b))|$

$= \binom{a+b-1}{a}$



Reflection is reversible

Reflect first segment in diagonal

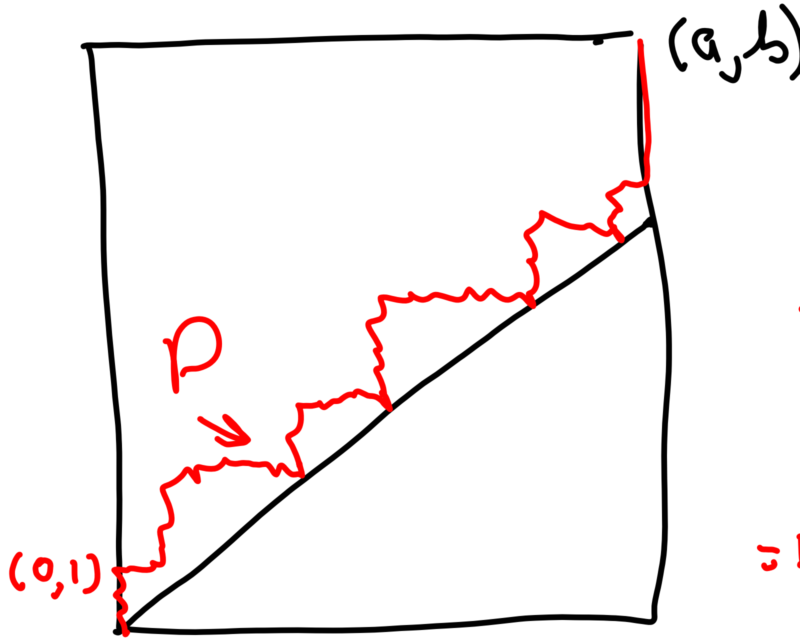


of P = # of P' = $\binom{a+b-1}{a-1}$

So

$$\left|\mathrm{PATHS}_{>}(a,b)\right| = \binom{a+b-1}{a} - \binom{a+b-1}{a-1} = \frac{b-a}{a+b}\binom{a+b}{a}$$

$PATHS_{\geq}(a,b)$

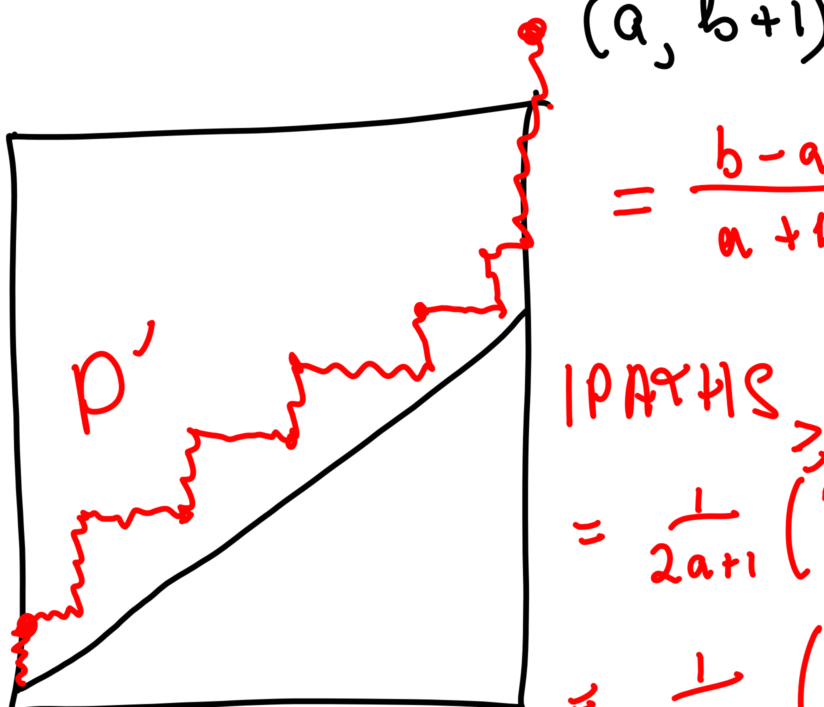


$\# p =$

$\# p'$

$= \# \, PATHS_{>}(a, b+1)$

Ballot Problem



$$= \frac{b-a+1}{a+b+1} \binom{a+b+1}{a}$$

$$|PATHS_{\geq}(a,a)| = \frac{1}{2a+1} \binom{2a+1}{a} = \frac{1}{a+1} \binom{2a}{a}$$

Catalan number