8/25/10

Binomial Coefficients.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

n choose k

X be a finite set

$$\binom{X}{k} = \{ S \subseteq X : |S| = k \}$$

Thm

$$\left| \binom{X}{k} \right| = \binom{|X|}{k}$$

$n = |X|$.

$k! \cdot \left| \binom{X}{k} \right|$ = # k-sequences of distinct elements of X.

$= n(n-1) \cdots (n-k+1)$.

How many sequences of positive integer $i_1, i_2, \dots, i_n$ are there such that:

$$i_1 + i_2 + \dots + i_n = m$$

Let this be $S(m,n)$

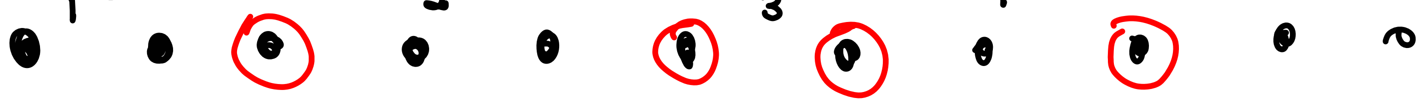
<u>Thm</u>

$$|S(m,n)| = \binom{m+n-1}{n-1}$$

$n-1$ ○

$m=7$
$n=5$

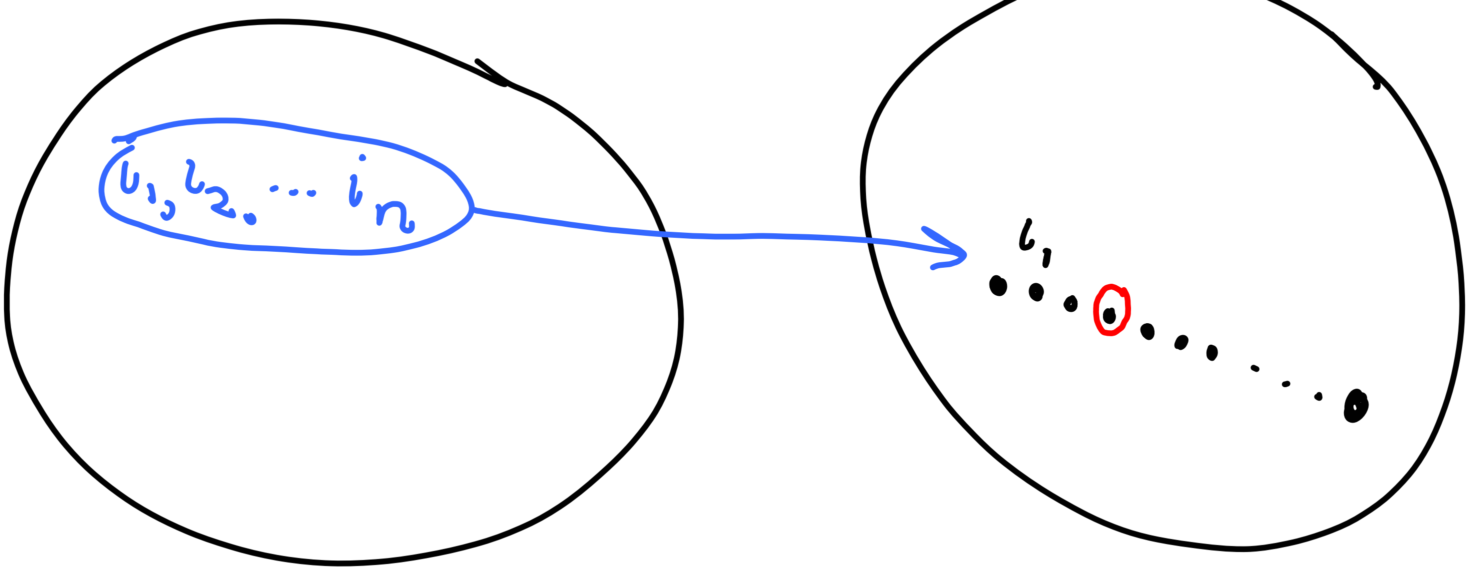
$i_1=2$ $i_2=2$ $i_3=0$ $i_4=1$ $i_5=2$



$m+n-1$ •'s

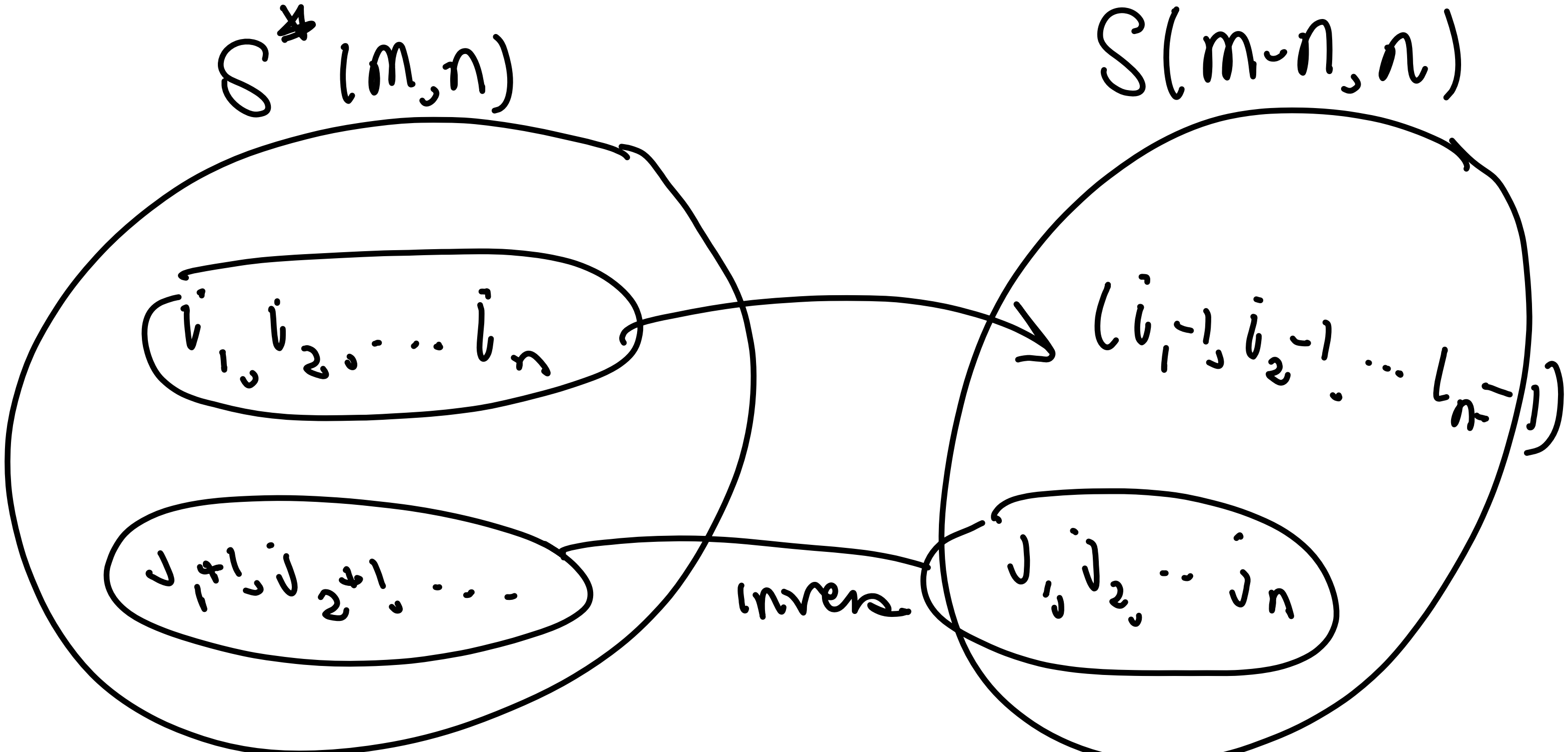
Sequences

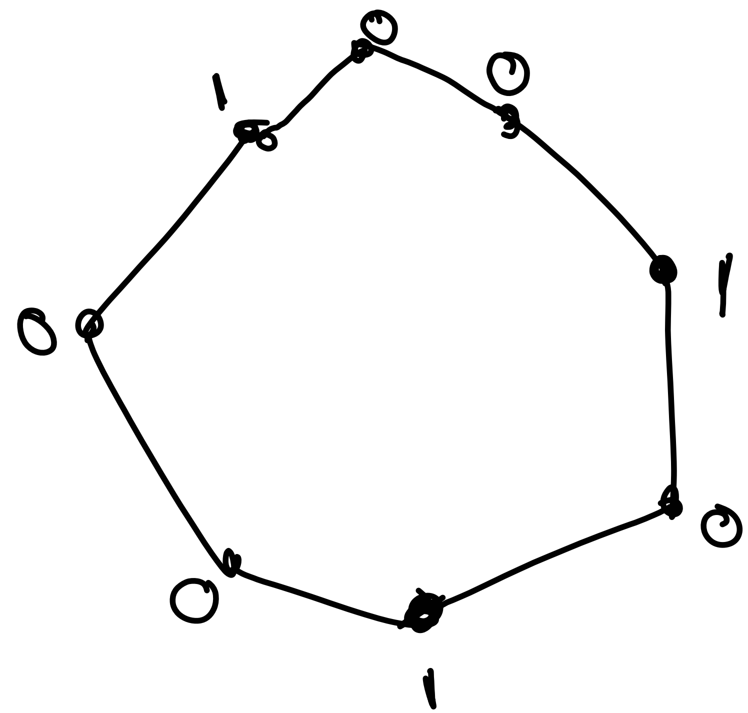
choices of ○



$S^*(m,n)$ = # ways, such that

$i_1, i_2, \dots i_n \geq 1$

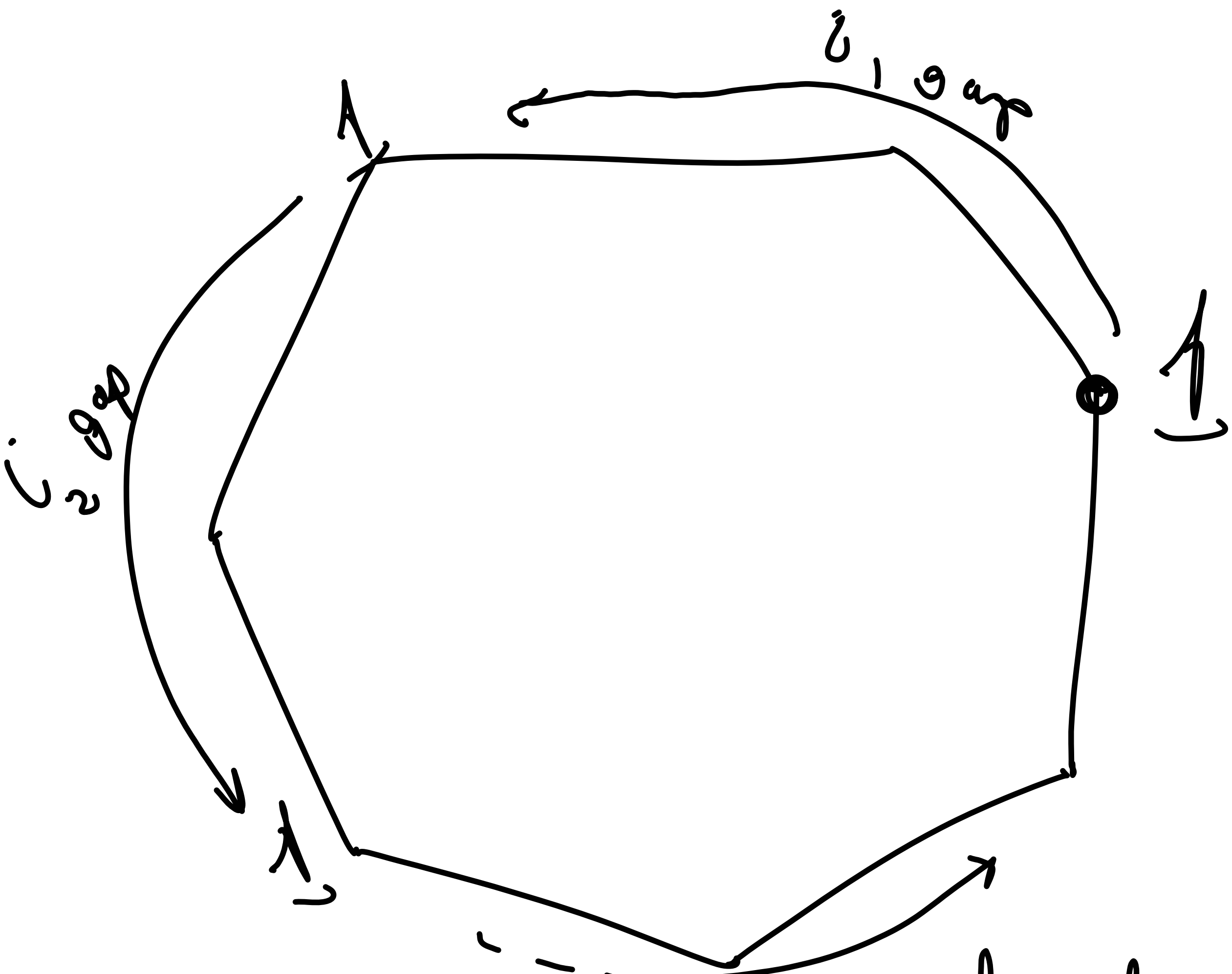




Polygon is fixed to plane – no rotation.

\# of patterns = ?

(1) Choose a vertex of polygon in n ways and place a 1 there



How many ways of finishing?

choose $k-1$ more 1's

$i_1 + i_2 + \dots + i_k = n - k$ & $i_1, i_2, \dots i_k \geq 1$

choices for the 1's =

choices for the $i_1, \ldots, i_{k-1}$

$$= \binom{n-k-1}{k-1}$$

Total choices $= n \times \binom{n-k-1}{k-1} \times \frac{1}{k}$

Binomial Identities

$$\binom{n}{r} = \binom{n}{n-r}$$

(i) Algerbraically $\quad \dfrac{n!}{r!\,(n-r)!} = \dfrac{n!}{(n-r)!\,(n-(n-r))!}$

or

(ii) Choosing to include

$\equiv$

choosing $n-r$ to exclude

Pascal's Triangle

$$\underbrace{\binom{n}{k}}_{\substack{\#\ (k+1)\text{-subsets} \\ \text{of } [n+1] \text{ that} \\ \text{contain } n+1}} + \underbrace{\binom{n}{k+1}}_{\text{exclude } n+1} = \underset{\substack{\text{all } k+1 \\ \text{subsets of} \\ [n+1]}}{\binom{n+1}{k+1}}$$

Generalised Pascal's Triangle.

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Proof

(1) By induction on n.

Assume true for n and all $k \le n$

$$\sum_{m=k}^{n+1} \binom{m}{k} = \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k}$$

$$\Big\downarrow \text{induction}$$

$$\binom{n+1}{k+1} + \binom{n+1}{k} = \binom{n+2}{k+1}.$$

(11)

$$\binom{k}{k} + \binom{k+1}{k} + \dots \binom{m}{k} \dots + \binom{n}{k} = \binom{n+1}{k+1}$$

counts all sets whose largest element is m+1

counts all k+1 sets in [n+1]