11\1\10

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

Proof

By induction on $k+l$.

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

$$\leq \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2}$$

$$= \binom{k+l-2}{k-1}.$$

In particula

$$R(k,k) \le \binom{2k-2}{k-1}$$

$$\le 4^k$$

Not much better is known

No proof that

$$R(k,k) \le (3.999999)^k$$

$$R(k,k) > 2^{k/2}$$

$$\sqrt{2} \leq R(k,k)^{1/k} \leq 4$$

Let $n = 2^{k/2}$. We have to show that there is a 2-coloring of the edges of K_n with no mono-chromatic clique.

Take a random coloring of K_n.

$\mathcal{E}_R$ = $\exists$ a red k-clique

$\mathcal{E}_B$ = $\exists$ a blue k-clique

To show

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1$$

Suppose that

$C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$

are the k-cliques of K_n.

$\mathcal{E}_{R,i}$ = C_i is Red

$\mathcal{E}_{B,i}$ = C_i is Blue

$$\mathcal{E}_R = \bigcup_{i=1}^{N} \mathcal{E}_{R,i}$$

$$
\begin{aligned}
\Pr(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B) \\
&= 2\Pr(\mathcal{E}_R) \\
&= 2\Pr\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \\
&\leq 2\sum_{j=1}^{N} \Pr(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}
\end{aligned}
$$

$$= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$\leq 2\,\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$< 2\,\frac{\left(2^{\frac{k}{2}}\right)^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$n<2^{\frac{k}{2}}$$

$$= \frac{2^{\frac{k^2}{2}-\frac{k(k-1)}{2}+1}}{k!}$$

$$= 2^{\frac{1}{2}k+1}/k! < 1$$

Ramsey's Theorem in General:

(1) We can have more than 2 colors.

4 colors

Dark Red, Light Red, Dark Blue, Light Blue

$$N = R\big(R(k,k),\ R(k,k)\big)$$

either Light Red k-clique / Dark Red k-clique ← either Red $R(k,k)$ clique or Blue $R(k,k)$ clique

(11) Colored edges of K_n

colored the ~~2~~-sets of $[n]$

r-sets of $[n]$

$n = 4$

$\{1,2,3\}$ $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$

Blue Green Green Blue

s-color, r-tples of $[n]$

$\Rightarrow \exists N(q_1, q_2, \dots, q_s ; r)$

$n \geq N(q_1, q_2, \ldots, q_s ; r)$

then in any s-coloring of the r-tuples of $[n]$

$\exists\, i$ & a set A_i of size q_i, such that all the r-tuples of A_i have color i.

Schur's Theorem

Color integers with k colors

$\exists$ a, b, c such that

$$a + b = c$$

and a, b, c have same color.

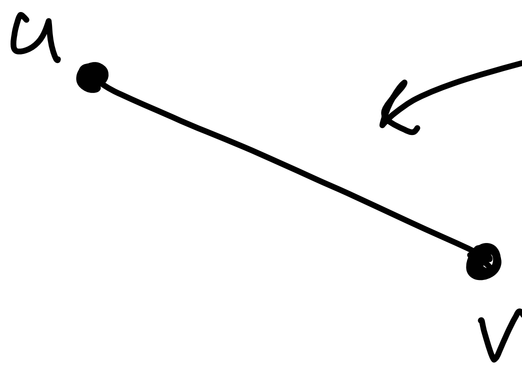
$$r_k = N(\underbrace{3, 3, \ldots, 3}_{k}; 2)$$

Color the edges of K_n with 2 colors & $\nexists$ monochromatic triangle.

Color $1, 2, \ldots, r_k$ with k-colors

$S_1, S_2, \ldots, S_k$ are the color classes.

We need to get a coloring of K_n $(n = r_k)$ out of this.

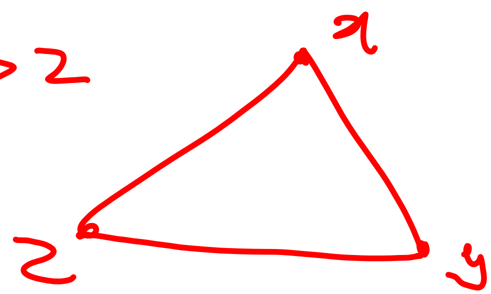


color $= i$

where $|u-v| \in S_i$

$\exists$ monochromatic triangle

$x > y > z$



$$\exists i: \quad \begin{array}{l} x-y \in S_i \\ y-z \in S_i \\ \hline x-z \in S_i \end{array}$$