11\12\10

Legal moves

(i) Can remove any even number of chips. Except when $k=2$. Can't take whole pile

(ii) Can remove whole pile if it is odd.

Terminal positions: 0 & 2

Grundy number:

$k =$	0	1	2	3	4	5	6	7	8	9	10
$g(k) =$	0	1	0	2	1	3	2	4	3	5	

It would seem that

$$g(2k) = k-1 \quad \& \quad g(2k-1) = k \quad ??$$

Prove this by induction.

True for small $k \leq 4$

Inductive Step:

$$\underset{k-1}{g(2k)} = \operatorname{mex}\{\underset{k-2}{g(2k-2)}, \underset{k-3}{g(2k-4)}, \ldots \underset{0}{g(2)}\}$$

$$\underset{2k}{g(2k-1)} = \operatorname{mex}\{\underset{2k-1}{g(2k-3)}, \underset{2k-2}{g(2k-5)}, \ldots \underset{1}{g(1)}, \underset{0}{g(0)}\}$$

Grundy number for the sums of games.

$G_1, G_2, \dots, G_p$ are games

$X_1, X_2, \dots, X_p$ are positions

$g_1(x_1) \quad g_2(x_2) \quad g_p(x_p)$ are grundy #s

claim

$$g(x_1, x_2, \dots, x_p) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

Bitwise sum.

$a = a_m a_{m-1} \cdots a_1 a_0$ in binary

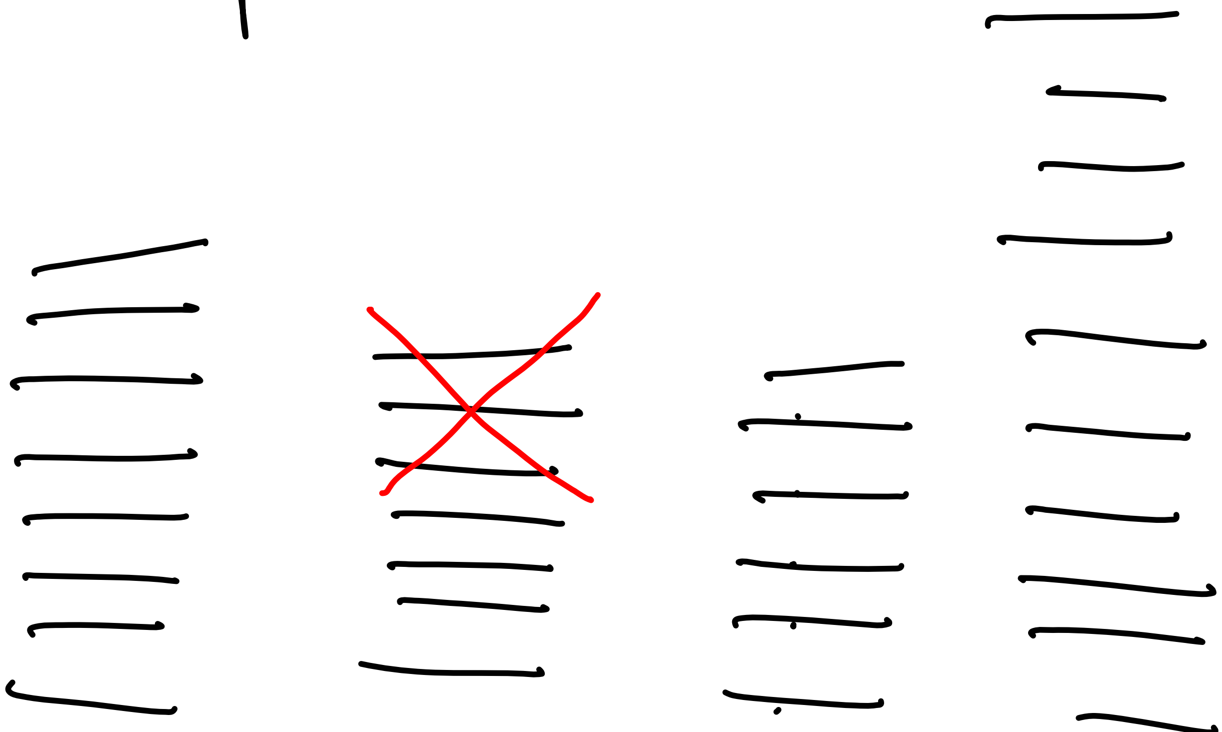
$b = b_m b_{m-1} \cdots b_1 b_0$ in binary

$a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ in binary

$$c_i = \begin{cases} 0 & a_i = b_i \\ 1 & a_i \neq b_i \end{cases}$$

Exclusive OR

Example



8 7 6 10

N position

$1000 \oplus 0111 \oplus 0110 \oplus 1010 = 0011$

0100 0000

Winning move – reduce 7 to 4

Claim $g(x_1, x_2, \dots, x_p) = g_1(x_1) \oplus \dots \oplus g_p(x_p)$

Enough to prove for $p=2$.

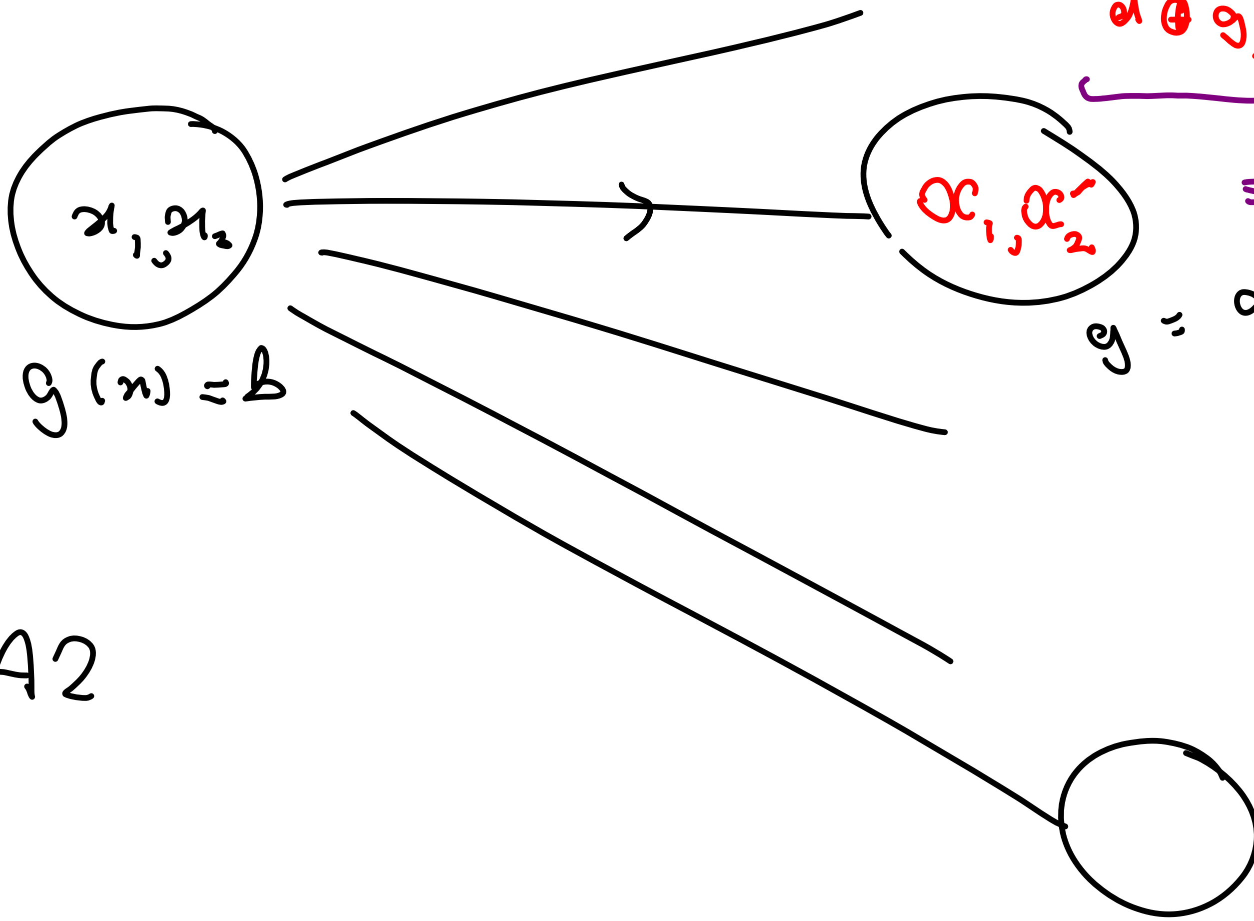
And then use induction

$$G_1 \oplus G_2 \oplus \dots + G_p$$

$$= (G_1 \oplus G_2 \oplus \dots \oplus G_{p-1}) \oplus G_p$$

Enough to show:

A1: $g(x) = b > a$



A2

A1

$a = d \oplus b$

$= d \oplus g_1(x_1) \oplus g_2(x_2)$

$d \oplus g_1(x_1) < g_1(x_1)$

or

$d \oplus g_2(x_2) < g_2(x_2)$

$\Rightarrow \exists x_2' : g_2(x_2') =$

and

$x_2 \rightarrow x_2'$

$$2^{k-1} \leq d < 2^{k}$$

$d :$ 0 0 0 0 1 . - - - .

k

$d_k = a_k \oplus b_k$

0 1

because $a < b$

$g_1(x_1)$ or $g_2(x_2)$ has a 1 in position k

$b = \; \oplus$

$d \oplus g_1(x_1) < g_1(x_1)$.