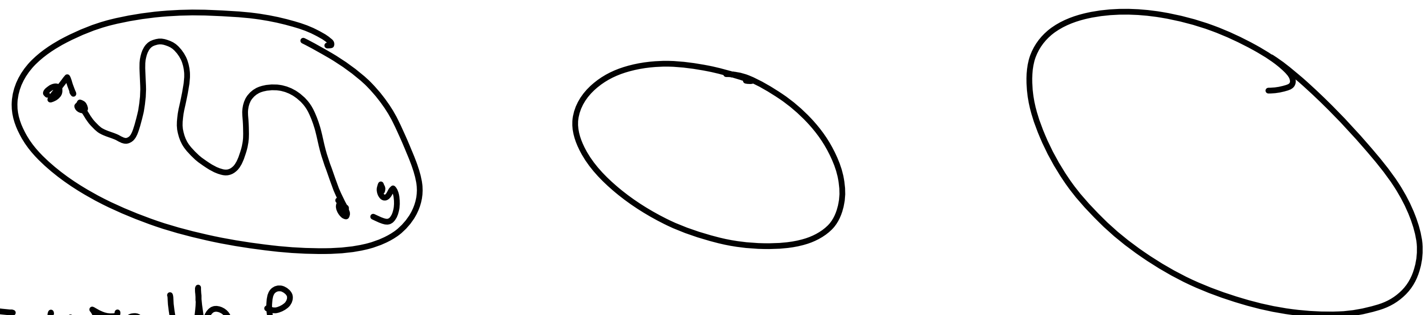
10/6/10

Connected Components



$\exists$ walk from x to y

$\Downarrow$

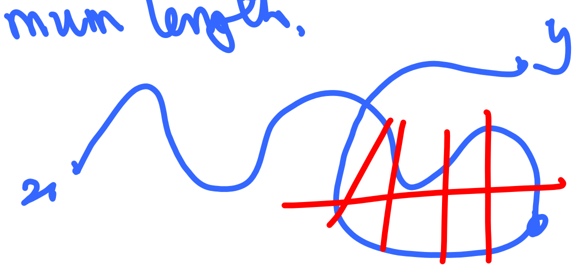
$\exists$ path from x to y :

walk

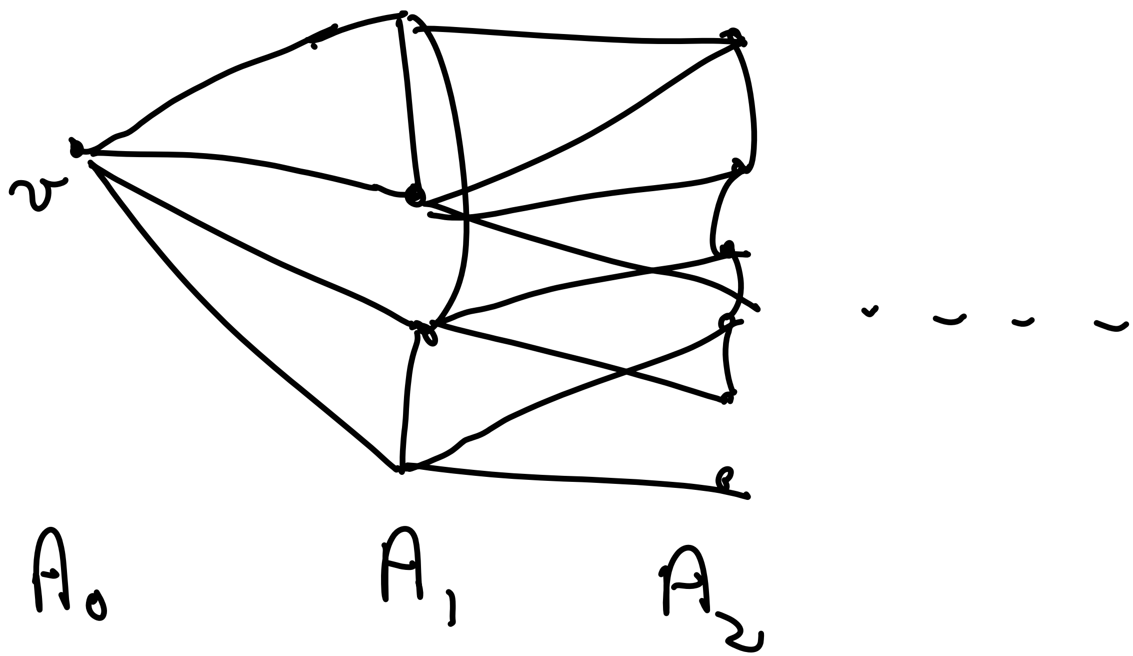
Let $\ell(W)$ = # edges in W.

Let P be a walk from x to y of minimum length.

<u>P is path</u>

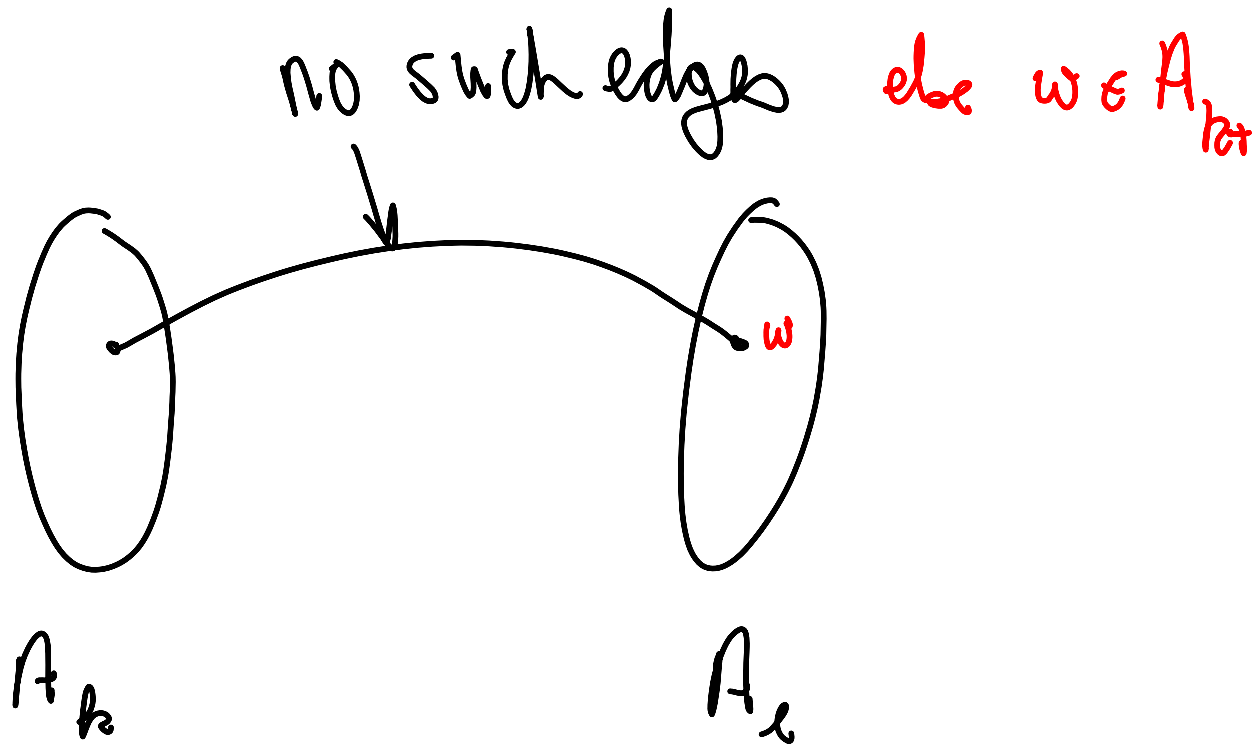


Breadth First Search BFS



$$A_{k+1} = \{ w \notin A_0 \cup A_1 \cup \dots \cup A_k : \exists v \in A_k \text{ with } vw \in E(G) \}$$

$$A_t = \{ w : \text{distance } v \text{ to } w = t \}$$



$\ell \geq k+2$

One can use BFS to find the components of G. Start with a vertex, find component containing v. Then do BFS from any vertex not found. . . .

Characterisation of bipartite graph.

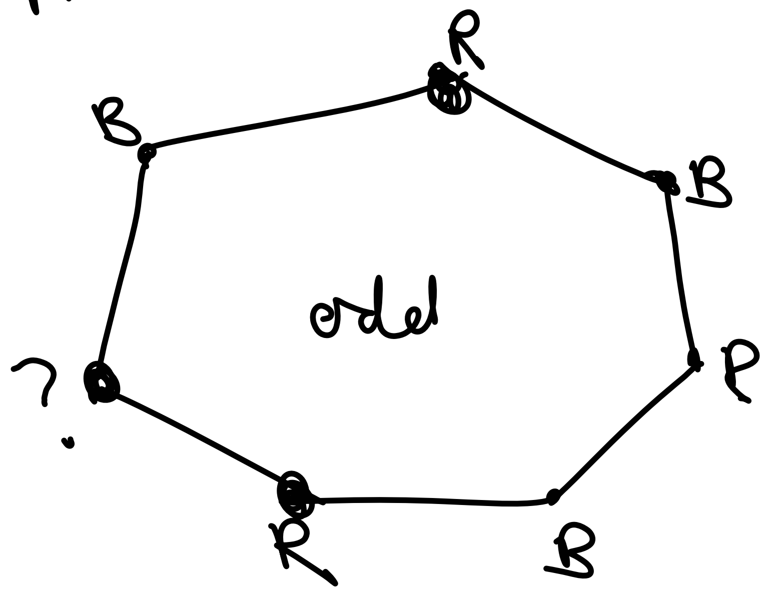
What are the smallest non-bipartite graphs?

<u>Theorem</u>

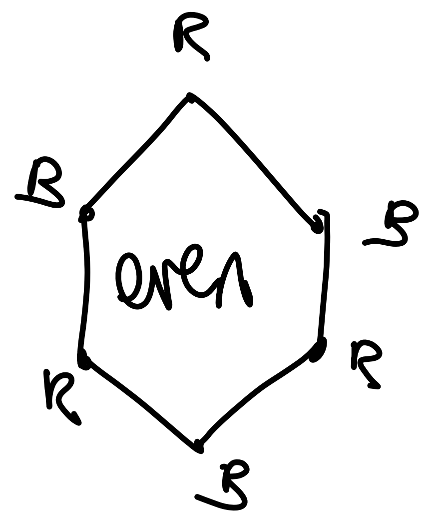
G is bipartite iff G has no cycles of odd length

Proof

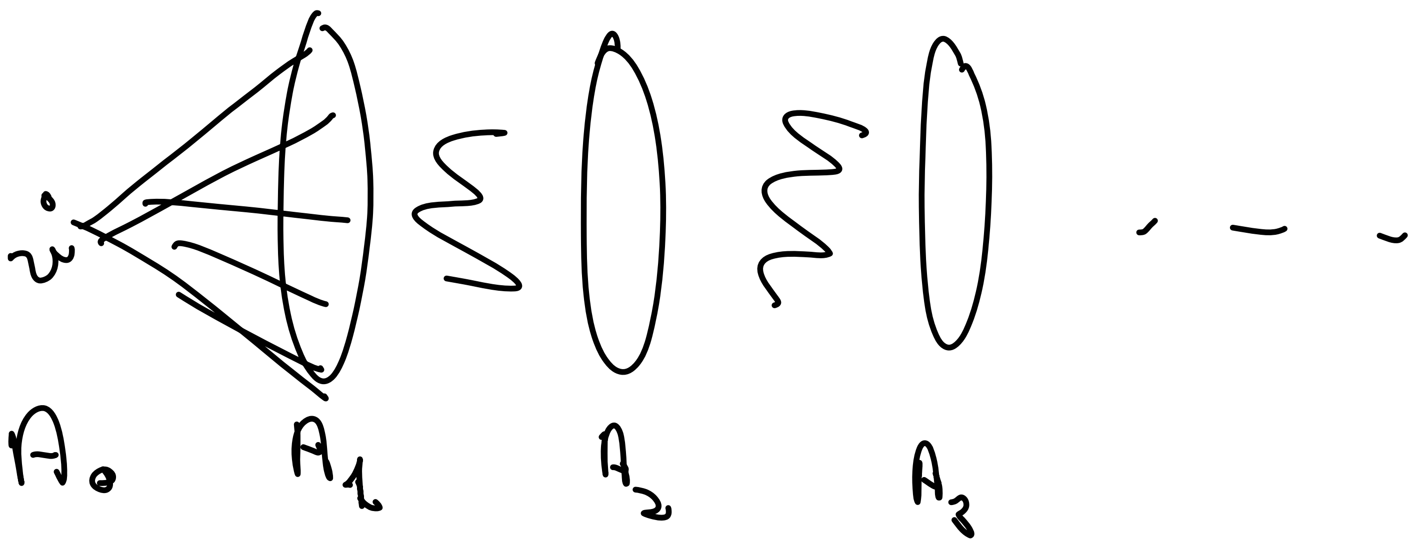
Suppose G contains an odd cycle



Bipartite $\equiv$ 2-colorable



Suppose there are no odd cycles.



$R = A_0 \cup A_2 \cup A_4 \cup \dots$

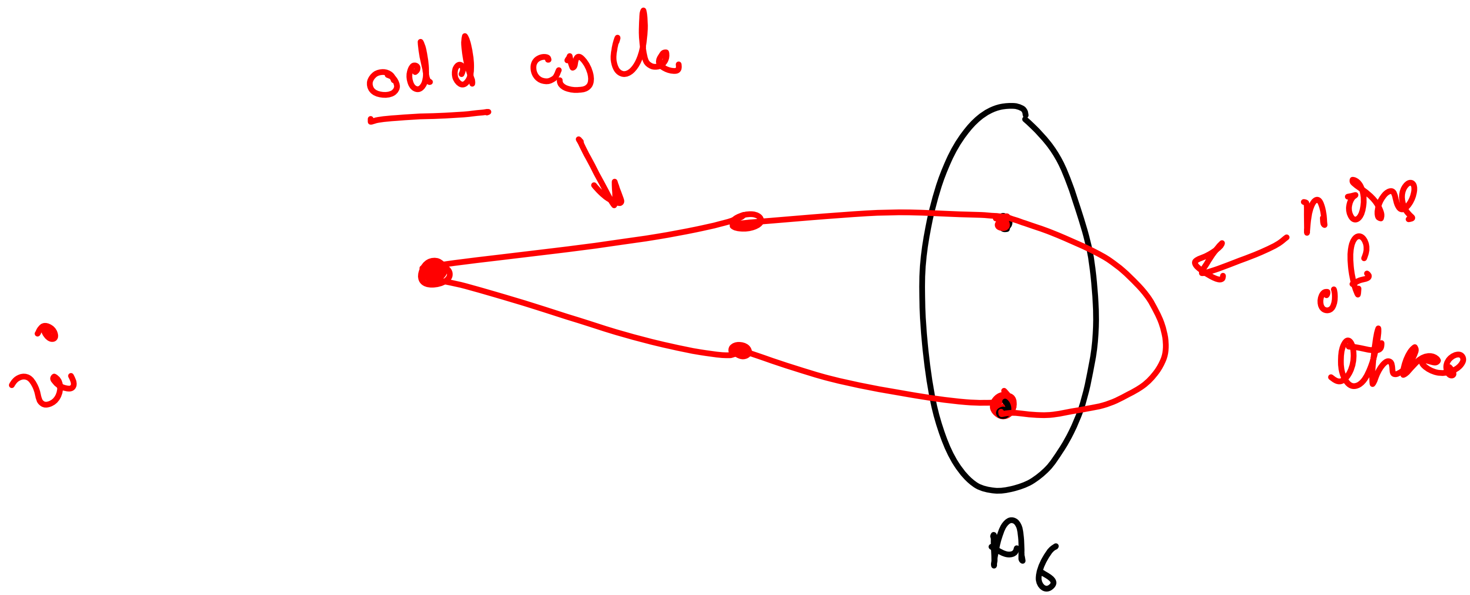
$B = A_1 \cup A_3 \cup \dots$

Claim is that all edge go from R to B

?? R – R edges

A_0 A_2 A_4 A_6

ruled out already on p3

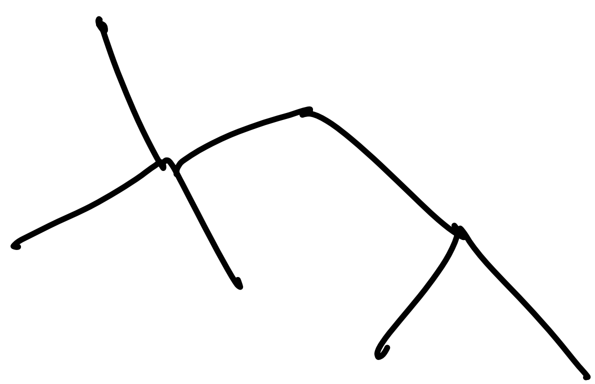


Trees

A tree is a graph which is

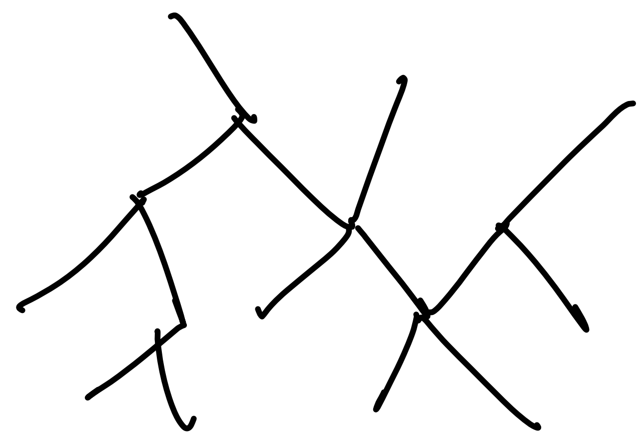
(a) Connected

(b) acyclic (no cycles)



8 vertices
7 edges

8 vertices
7 edges



16 vertices
15 edges

Lemma

G has components $C_1, C_2, \ldots, C_r$

$e = (u, v) \notin E(G)$

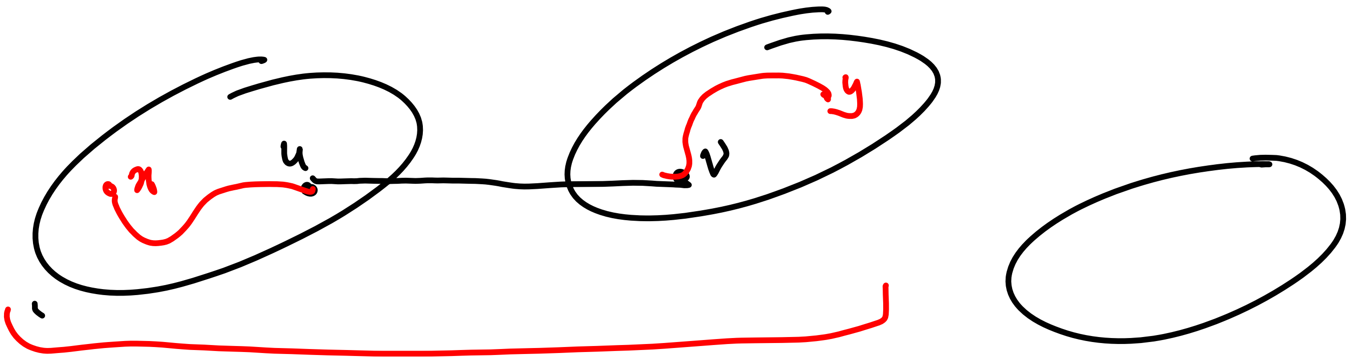
$G' = G + e$

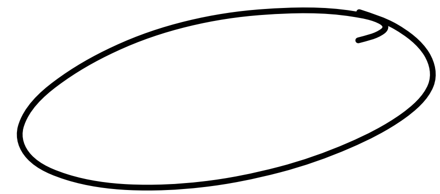
(a) If u, v are in different components of G then G' has one less component than G

(b) If u, v are in the same component then

components of G' = # components of G.

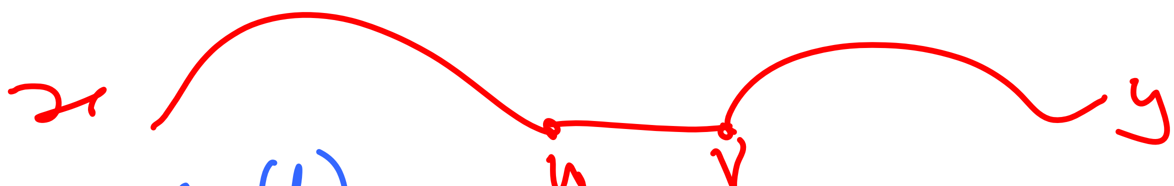
(a)





This become a single component

If x become connected to y after adding (u,v) then any path looks like



Also prove (b).

Lemma

Now assume G is acyclic.

Suppose we add edge $(u,v) \notin E(G)$.

Components of G are $C_1, C_2, \ldots C_k$

vertices $= n$.

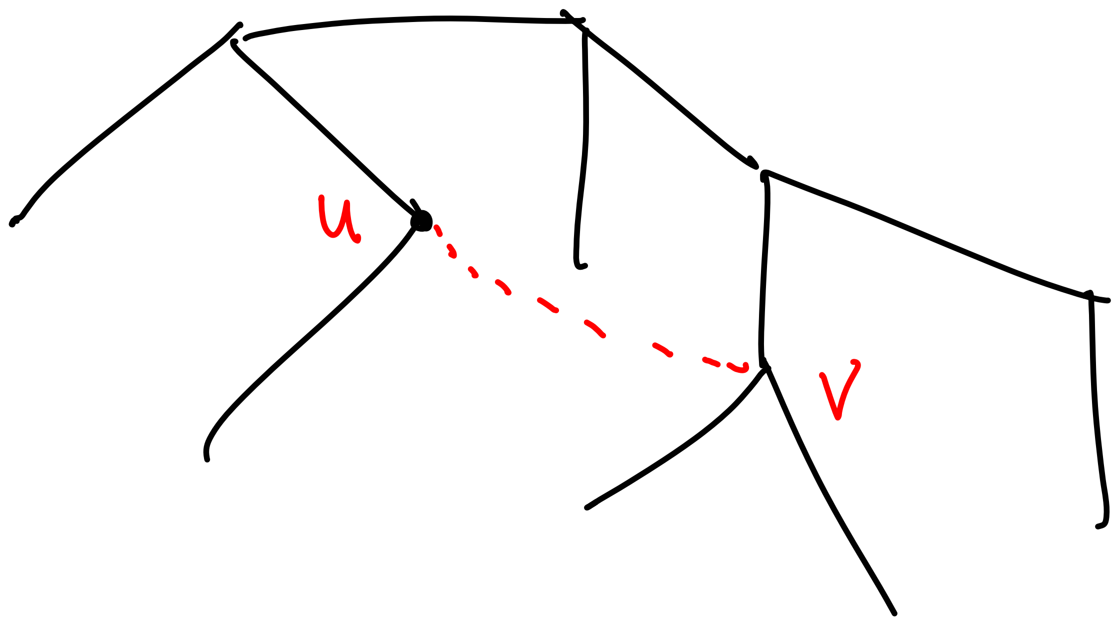
$u \in C_i$, $v \in C_j$

(a) $i = j \Longrightarrow$ $G+e$ has a cycle

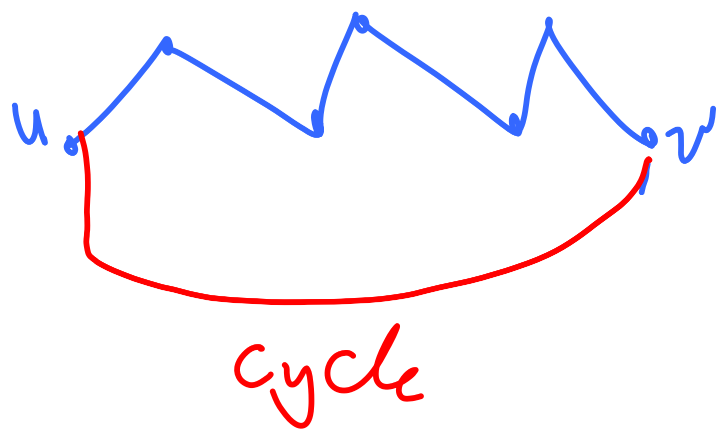
(b) $i \neq j \Longrightarrow$ $G+e$ has no cycles and $k-1$ components

(c) G has $n-k$ edges

(a)

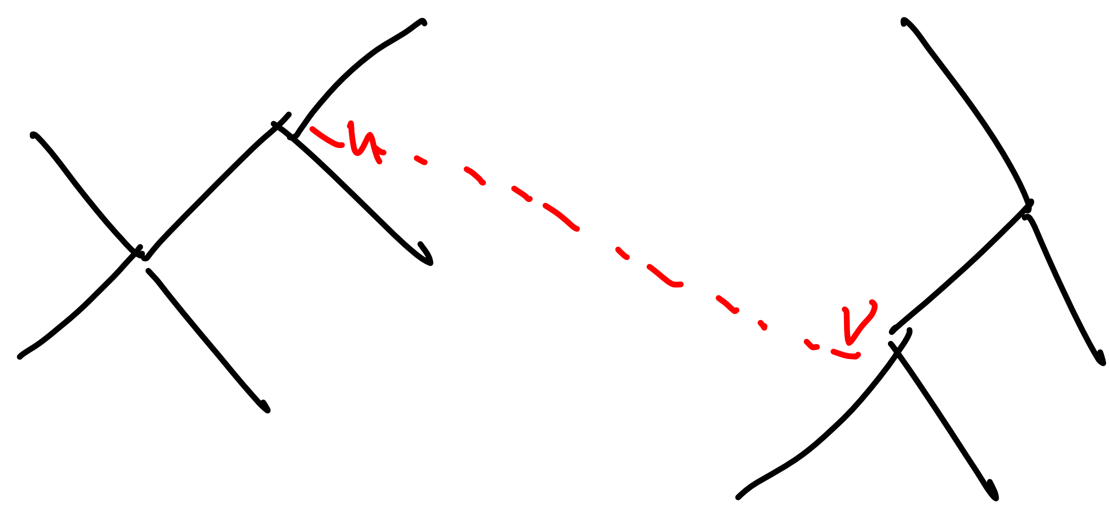


There is a path in G



components stays same.

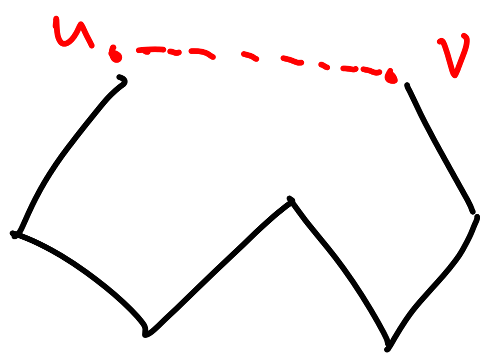
(b)



comps drops by one

no cycle

Suppose $\exists$ cycle



$\Rightarrow$ u,v in same component.

In summary if

G is acyclic then

(i) A cycle and same # of components

G+e has

or

(ii) is acyclic and has one less

(c) <u>k components</u>

$$E(G) = \{e_1, e_2, \dots, e_t\}$$

$$G_i = (V, \{e_1 \dots e_i\})$$

$$\text{So} \quad G_i = G_{i-1} + e_i$$

G_0 has n vertices and 0 edges

each time I add an edge, I do not make a cycle. Therefore I reduce # component by 1, Reduce from n to k

$\Rightarrow \exists\ n-k$ edges

Corollary

If Tree T has n vertices then it has $n-1$ edges.

$$\Downarrow$$

$\exists$ at least 2 vertices of degree 1.

$$\sum_{v \in T} d_T(v) = 2n-2$$