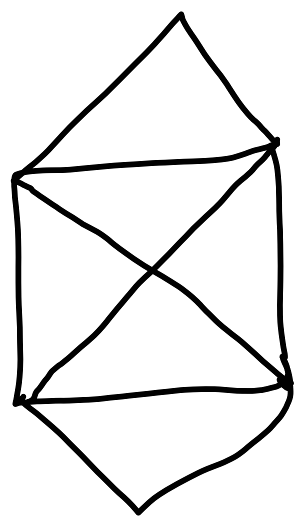
10\4\10

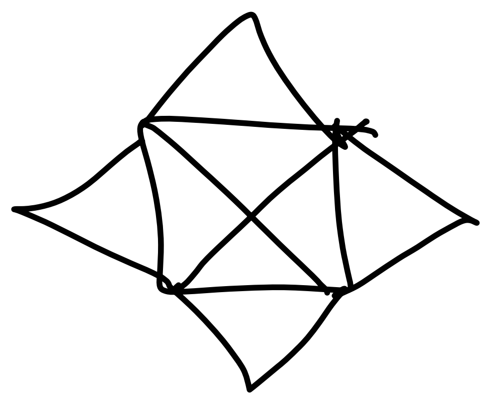
Graph - definition ✓

Digraph - definition ✓

Eulerian Graphs — graphs that can drawn without going over the same edge twice and without lifting pen from paper.

Eulerian





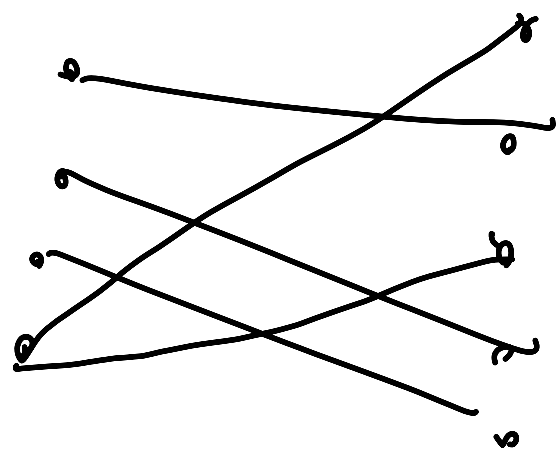
NO

Bipartite Graphs - Definition

G, Vertex set $A \cup B$, $A \cap B = \emptyset$

All edges are of the form (a, b)

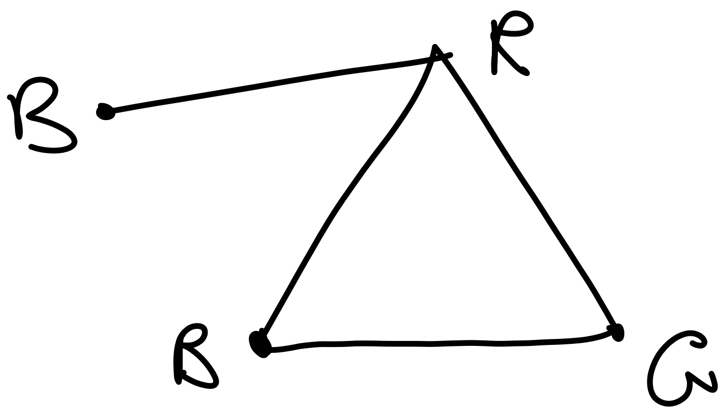
where $a \in A$, $b \in B$



Vertex Coloring

Graph G.

Color vertices so that neighboring vertices have different colors



chromatic number

$\chi(G)$ = minimum number of colors needed.

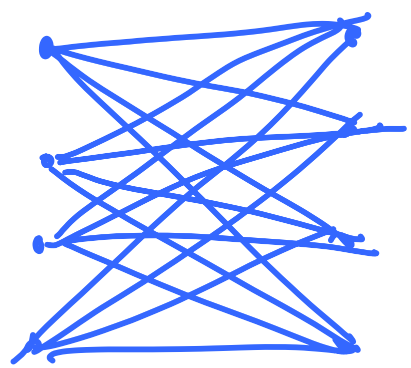
Bipartite graphs have chromatic number ≤ 2.

Sub-graph ✓

Isomorphism ✓

Complete Graphs

$K_{4,4}$

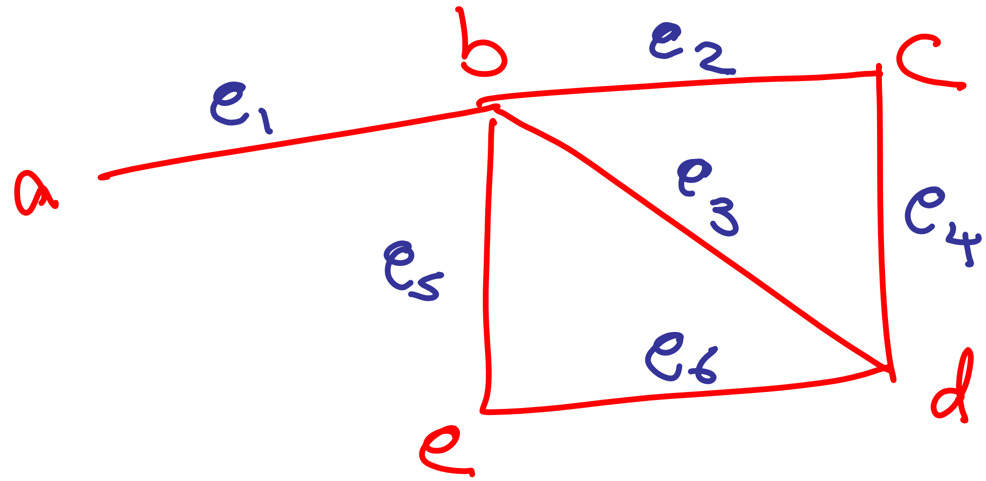


Degrees ✓

Matrices and Graphs

Incidence matrix M

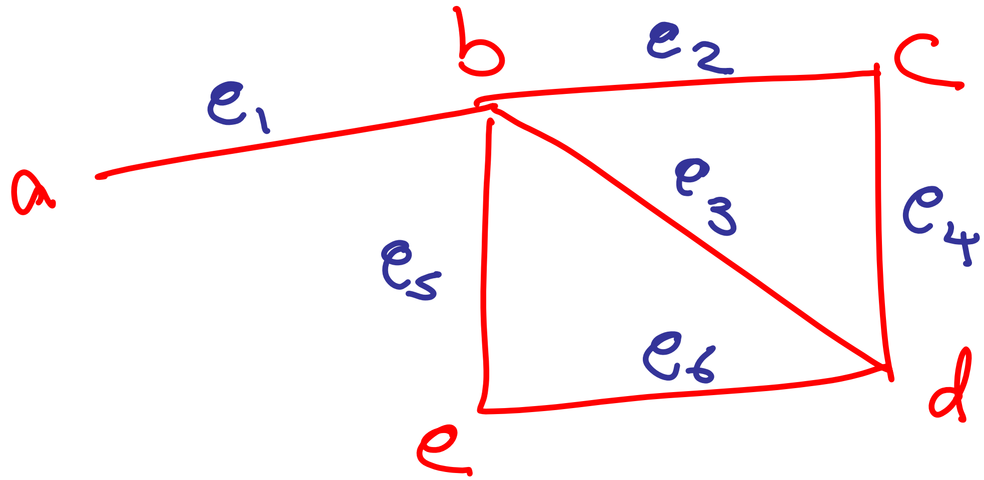
$$\begin{array}{cc} & E \\ V & \begin{bmatrix} & & \\ & 1_{v \in e} & \\ & & \end{bmatrix} \end{array}$$



$$
\begin{array}{c}
\\ a \\ b \\ c \\ d \\ e
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\left[\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & 1 & & 1 & \\
 & 1 & & 1 & & \\
 & & 1 & 1 & & 1 \\
 & & & & 1 & 1
\end{array}\right]
\end{array}
$$

Adjacency Matrix

$$\begin{array}{cc} V \diagdown V & v \\ u & \begin{bmatrix} \\ 1_{u,v \text{ is an edge}} \\ \\ \end{bmatrix} \end{array}$$



Adjacency matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Symmetric

Theorem

$$\sum_v d_G(v) = 2|E|.$$

$$M \qquad \begin{array}{cc} & e \\ v & \begin{bmatrix} \\ 1 \\ 1 \\ \end{bmatrix} \end{array} \leftarrow d_G(v)$$

$$\#1\text{'s in } M = \begin{cases} 2|E| & \text{columns} \\ \sum_v d_G(v) & \text{rows} \end{cases}$$

Corollary

In any graph, the number of vertices of odd degree, is even.

$$\underbrace{\sum_{v \text{ odd}} d_G(v)}_{\text{even}} = \underbrace{2|E|}_{\text{even}} - \underbrace{\sum_{v \text{ even}} d_G(v)}_{\text{even}}$$

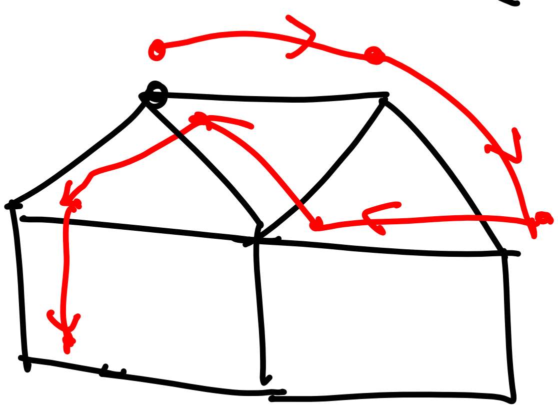
Paths and Walks

Walk = sequence of vertices

$v_1, v_2, \ldots, v_k$ length $= k-1$ # edges crossed

where $(v_i, v_{i+1}) \in E$

$1 \le i < k$

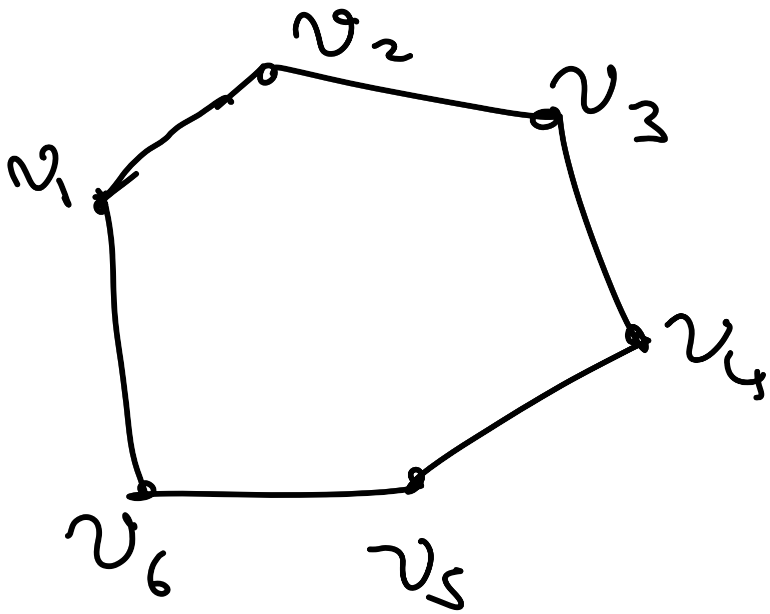


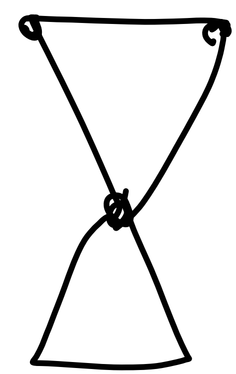
Path if walk visits vertex at most once.

Closed walk : if $v_1 = v_k$

Cycle : $v_1, v_2, \ldots v_k = v_1$

$v_i \neq v_j$





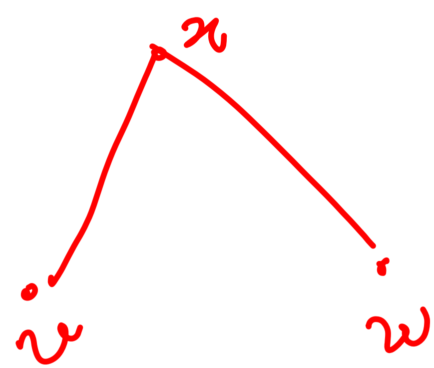
NOT A CYCLE

A = adjacency matrix of G.

$A(v,w)$ = # walks of length 1 from v to w.

$$A^2(v,w) = \sum_x A(v,x)\,A(x,w)$$

Counts

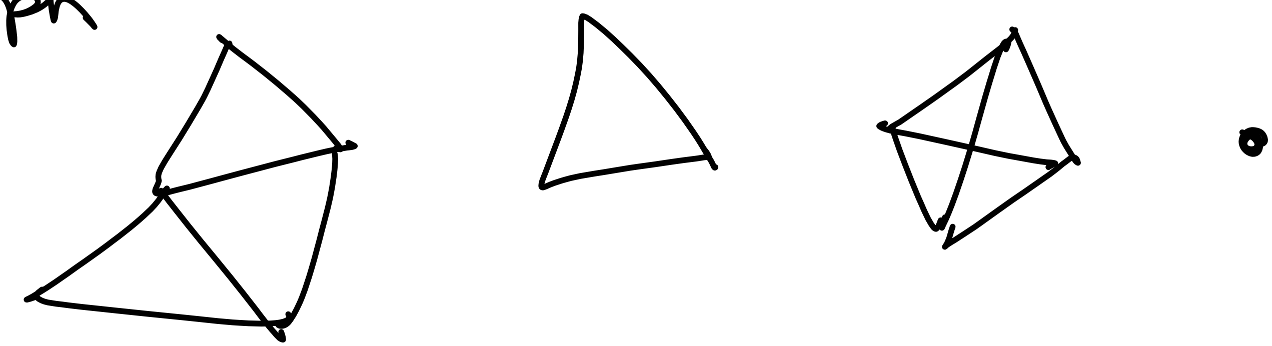


Proof by induction

$$A^{k+1}(v,w) = \sum_{x} A^{k}(v,x)\, A(x,w)$$

Connected Components

One Graph



4 pieces - components

Define a relation $\sim$ on V

$a \sim b$ iff $\exists$ walk a to b

(i) reflexive $a \sim a$

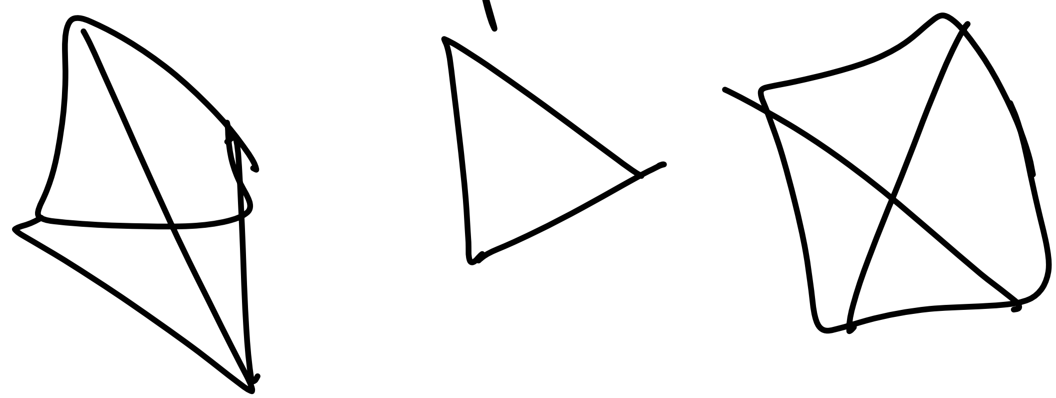
(ii) symmetric $a \sim b \Leftrightarrow b \sim a$

(iii) transitive a — b — c

Equivalence classes of $\sim$ are called the connected components.

Components

G



If a, b in same component, $\exists$ walk a to b

diff $\nexists$

If there is one component, then the graph is <u>connected</u>