10\22\10

# Hall's Theorem

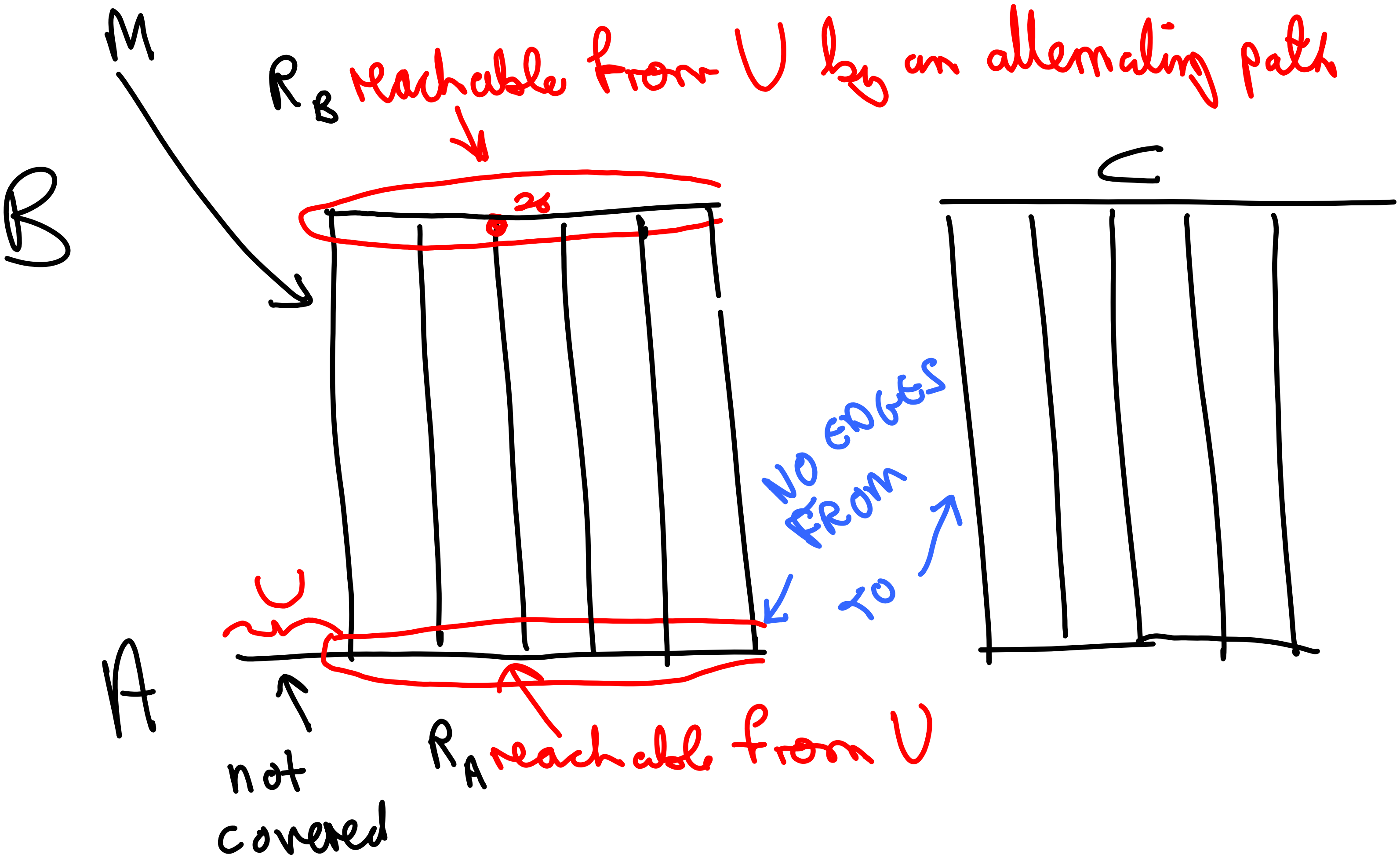
G contains a matching of A into B iff

$$|N(S)| \geq |S| \quad , \forall S \subseteq A$$

Suppose that $|N(S)| \geq |S|$, $\forall S \subseteq A$.

Suppose that there is no matching that covers $A$.

Suppose that $M$ is a maximum size matching.

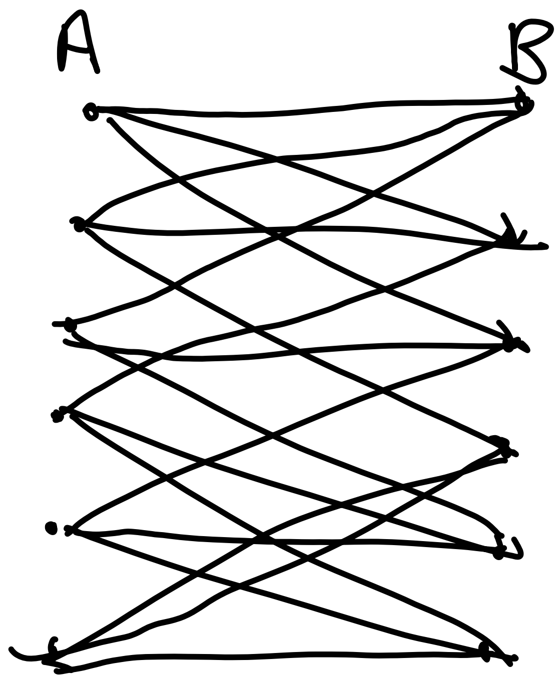




$S = U \cup R_A$ $\quad |N(S)| = |R_B| < |S|$, contradiction

# Marriage Theorem

$k=3$

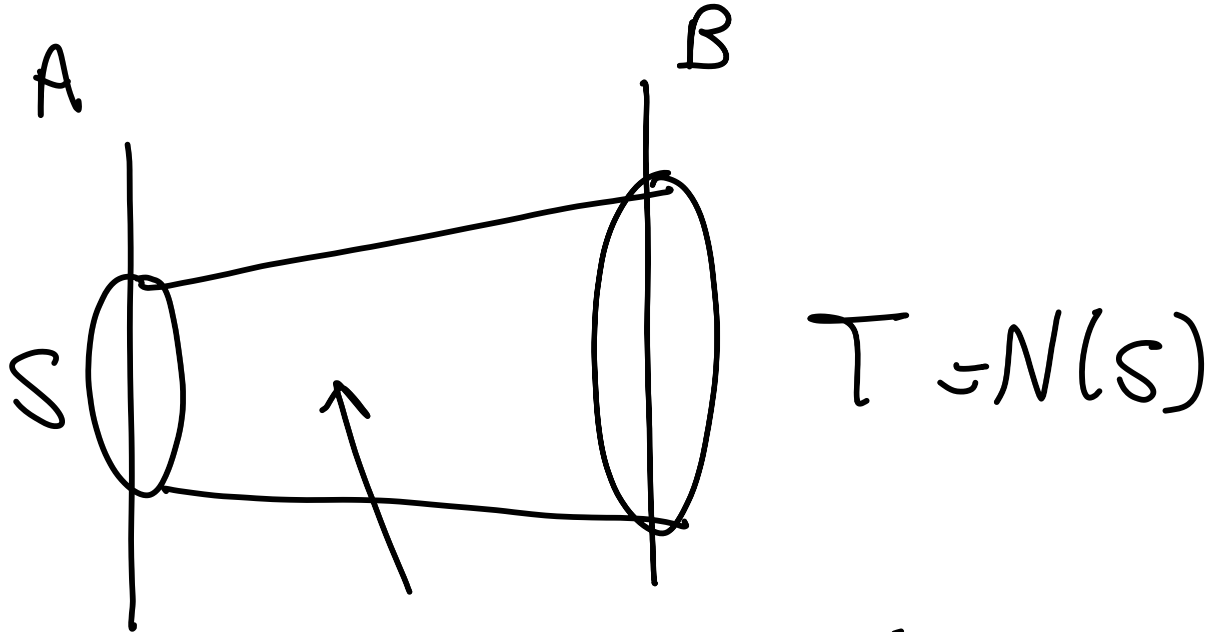



every vertex has degree 3.

more than one copy of an edge

$G$ is a $k$-regular bipartite (multi)graph

Claim: $G$ has a perfect matching




$m = \#$ of $S \to T$ edges

$$k|S| = m \leq k|T|$$

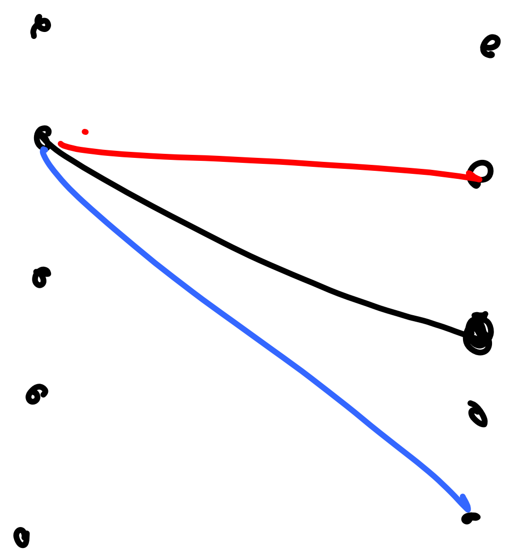
$$\Rightarrow |T| \geq |S|$$

Now apply Hall's Theorem. $[|A| = |B|$

## Corollary

If $G$ is a $k$-regular bipartite graph then you can color the edges of $G$ with $k$-colors so that no 2 edges incident with same vertex have the same color.

$k=3$



By induction on $k$.

True for $k=1$: a 1-regular graph is a perfect matching

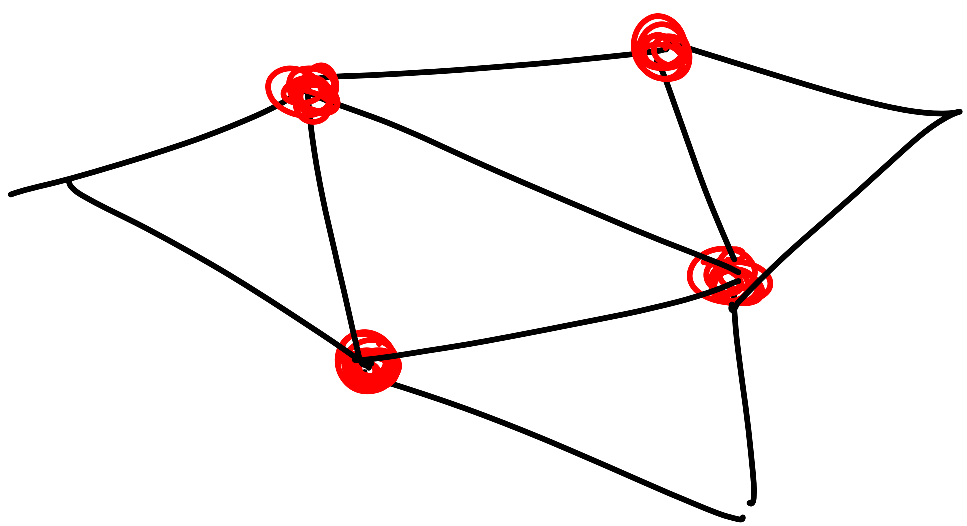
Assume true for $k-1$.

Marriage theorem says $\exists$ perfect matching $M$.

Color $M$ red & $G-M$ with $k-1$ colors.

# Edge Covers

$X \subseteq V$ is an edge cover if every edge contains a member of $X$



● is a cover.

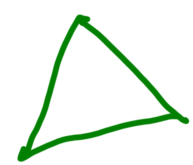
If $X$ is a cover and $M$ is a matching then $|X| \geq |M|$

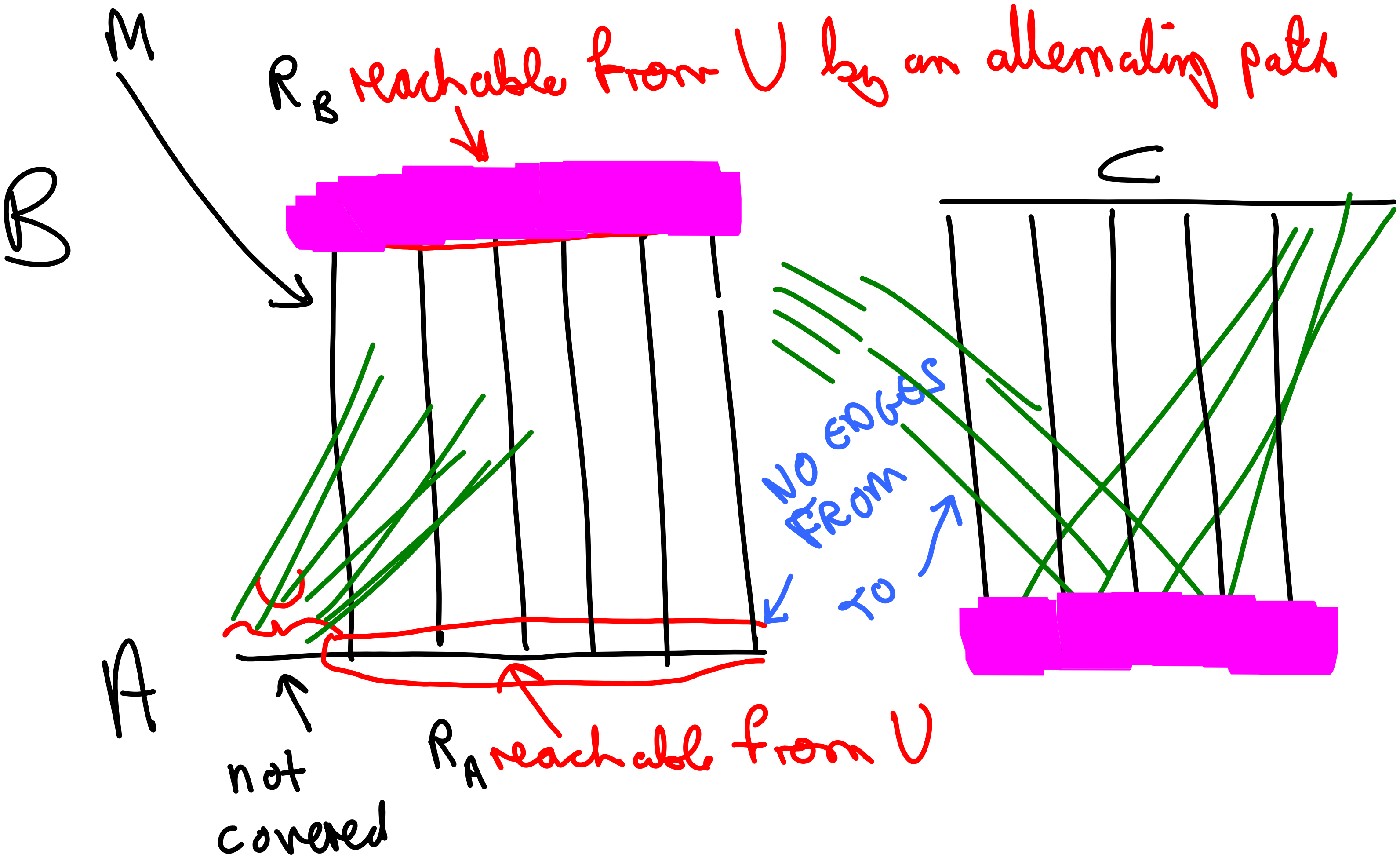
Each vertex in $X$ can cover at most one edge in $M$.

König's theorem:

If $G$ is bipartite then

$$\max_{M \text{ is a matching}} |M| = \min_{X \text{ is a cover}} |X|$$





M
$R_B$ reachable from U by an alternating path
B
C
NO EDGES FROM TO
A
not covered
$R_A$ reachable from U

column corr. vertex of B

row corr. to a vertex of A

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

König:

Max size of independent 1's = min # of lines needed to cross out every 1

A set of 1's are "independent" iff no two lie on same row or column.

A "line" is a row or column.