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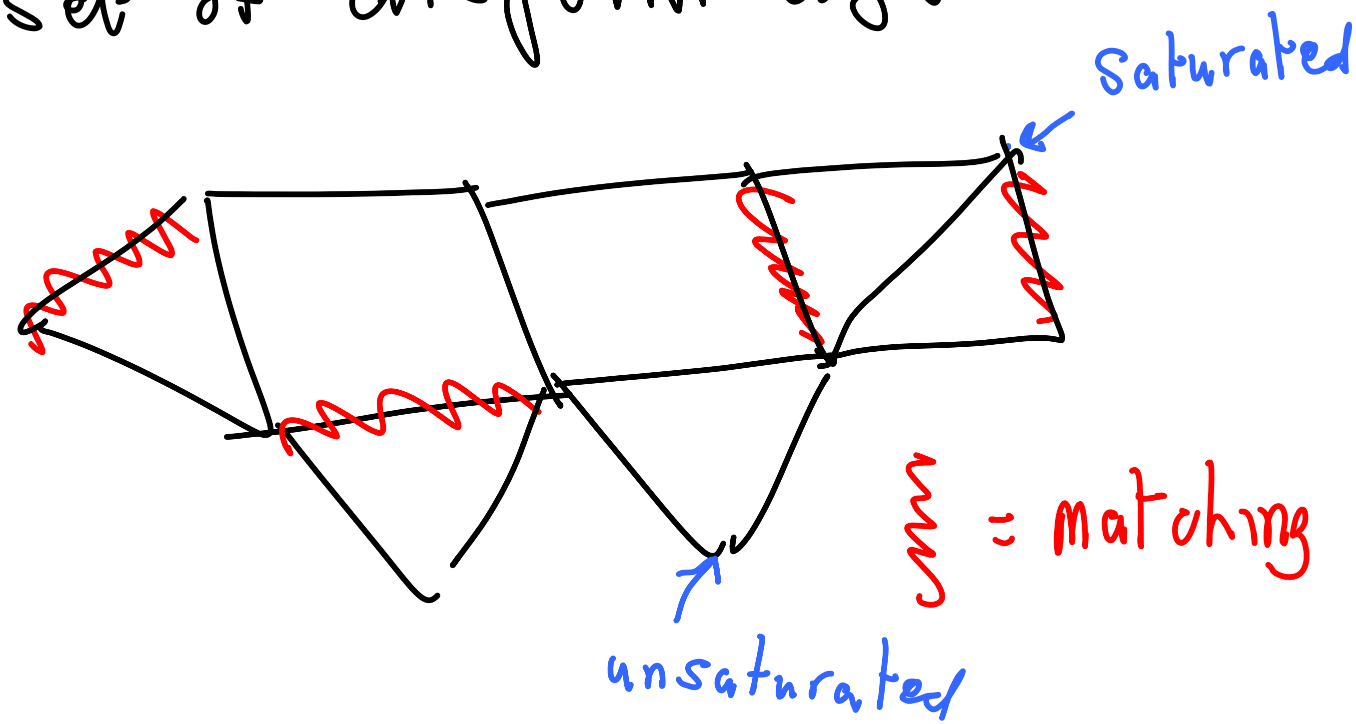
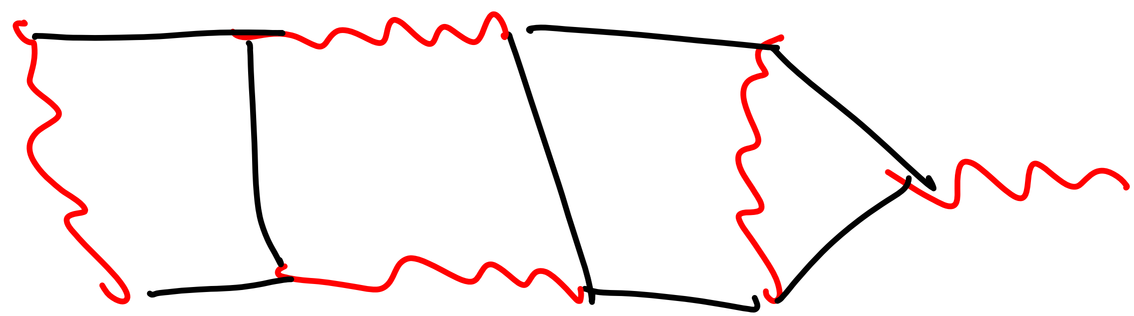
Matchings

A matching M of a a graph G is a set of disjoint edges



Question: what can one say about the size of the largest matching in G?

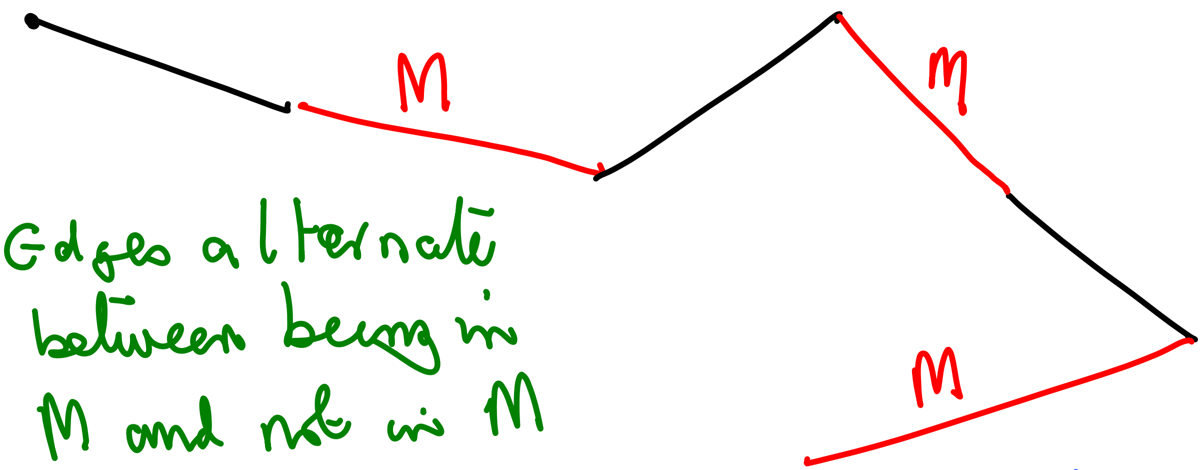
A matching is perfect if it saturates every vertex



Alternating Path

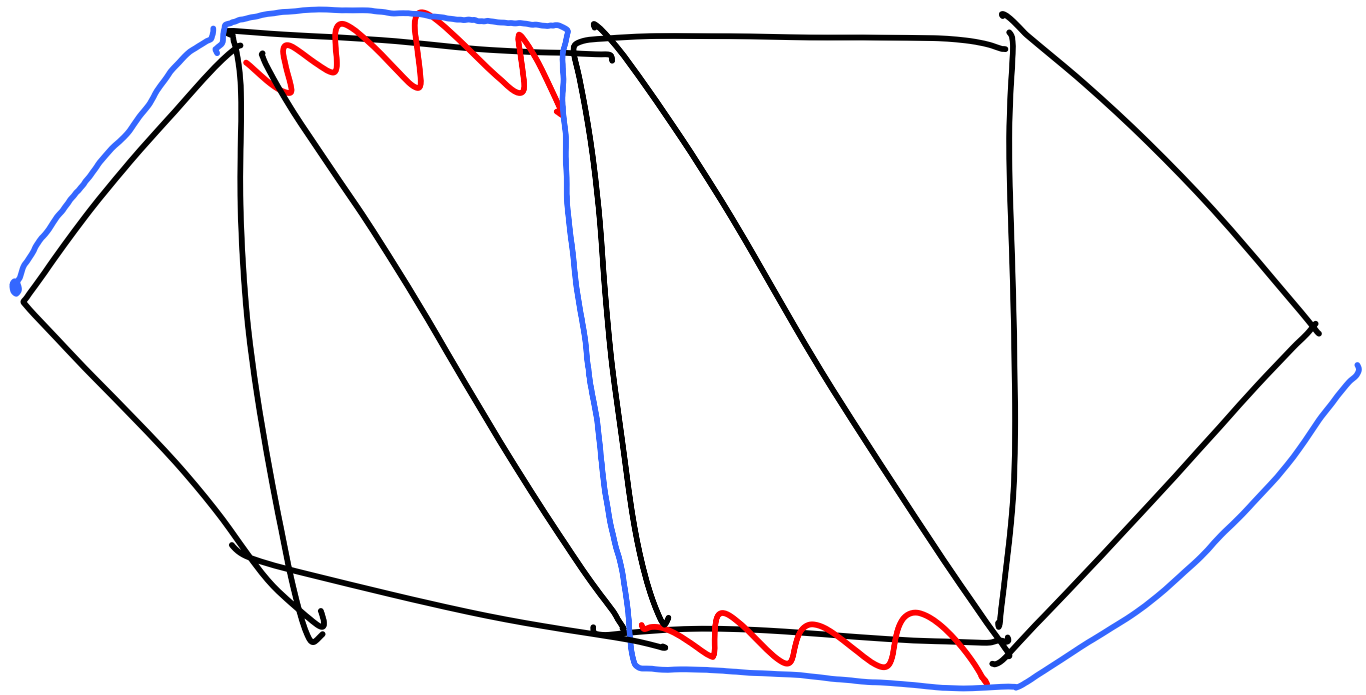
Given graph G.

Matching M



Edges alternate between being in M and not in M

An augmenting path is an alternating path joining 2 unsaturated vertices.

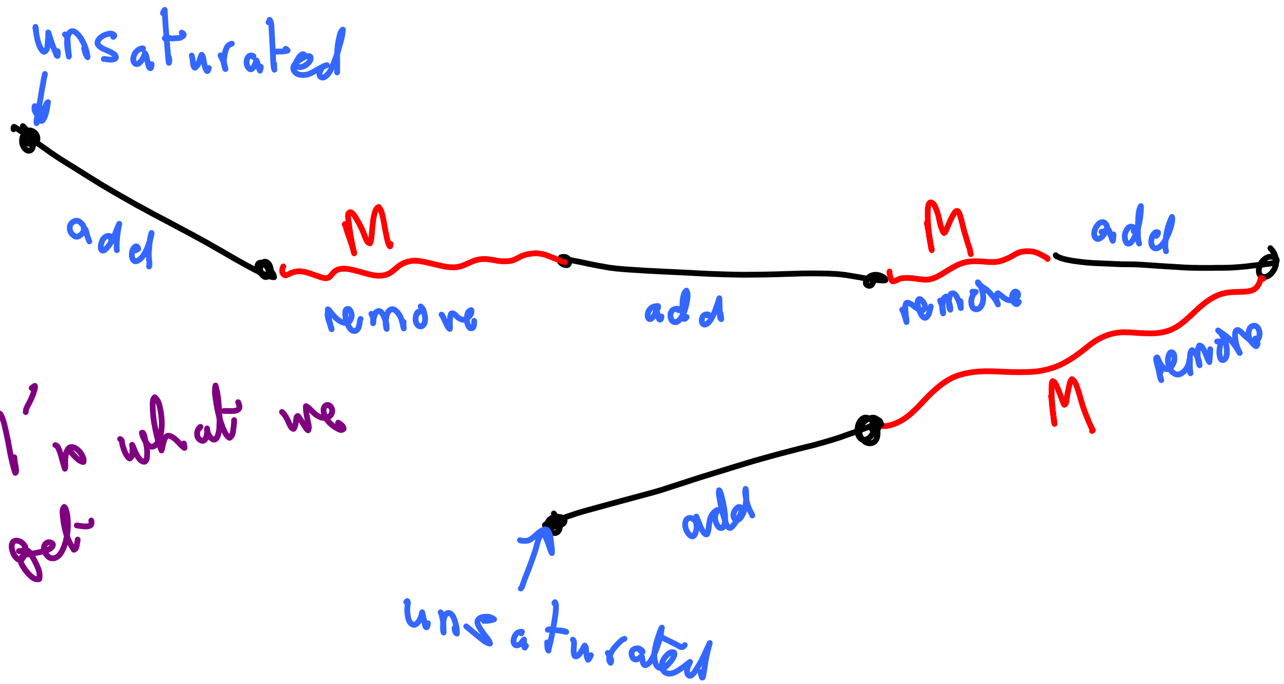


Blue path is augmenting

Thm

M is a <ins>maximum</ins> <ins>size</ins> matching iff there are no augmenting paths w.r.t. M.

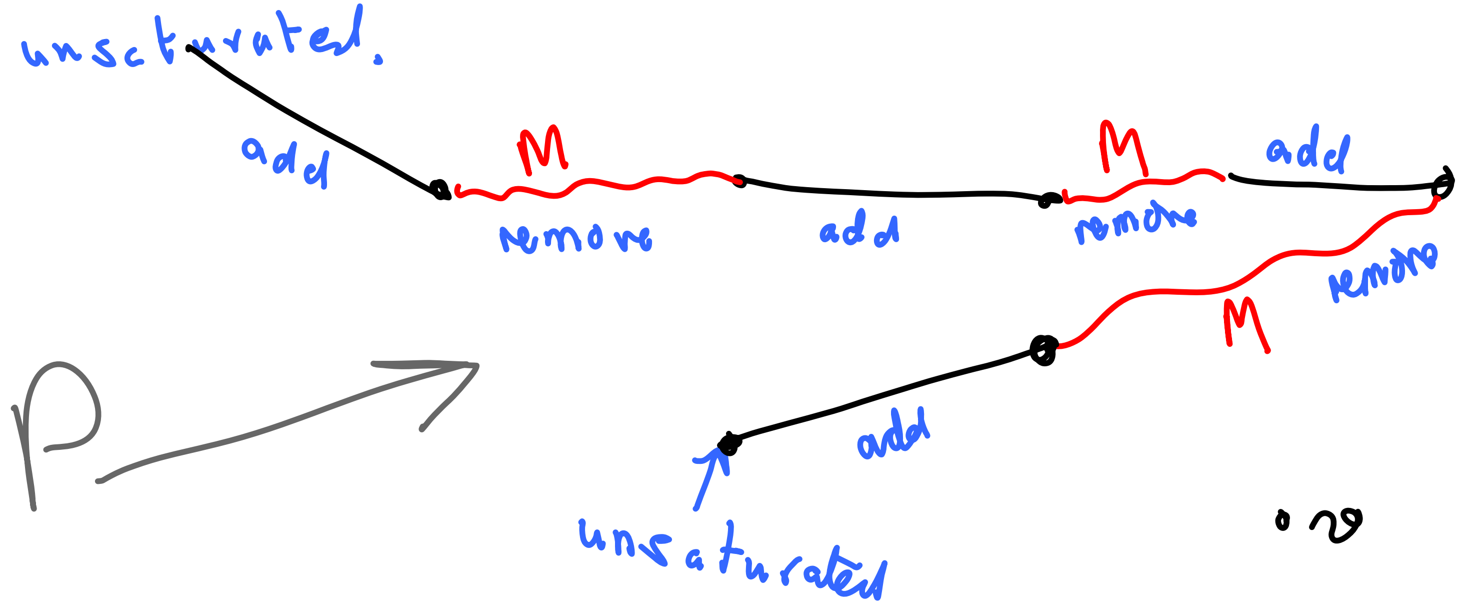
(1) Suppose there is an augmenting path



M′ is what we get

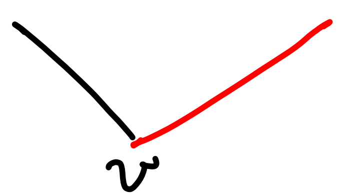
(i) One more black edge than red edge i.e. add an edge.

(ii) Do we still have a matching?

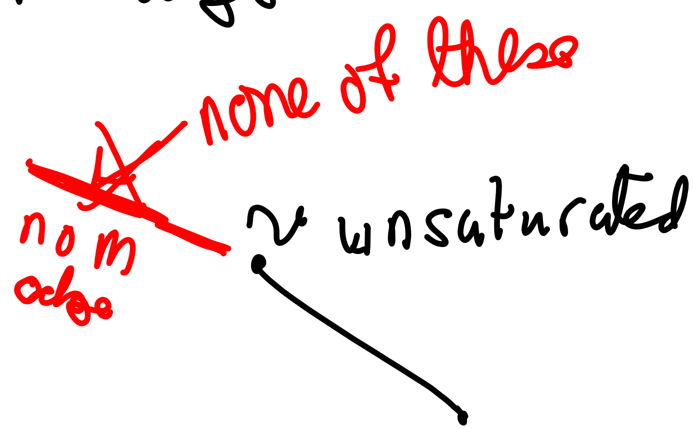


$v \notin P$: v meets same number of M' edge as M edge.

$v \in P$:



still meets one edge

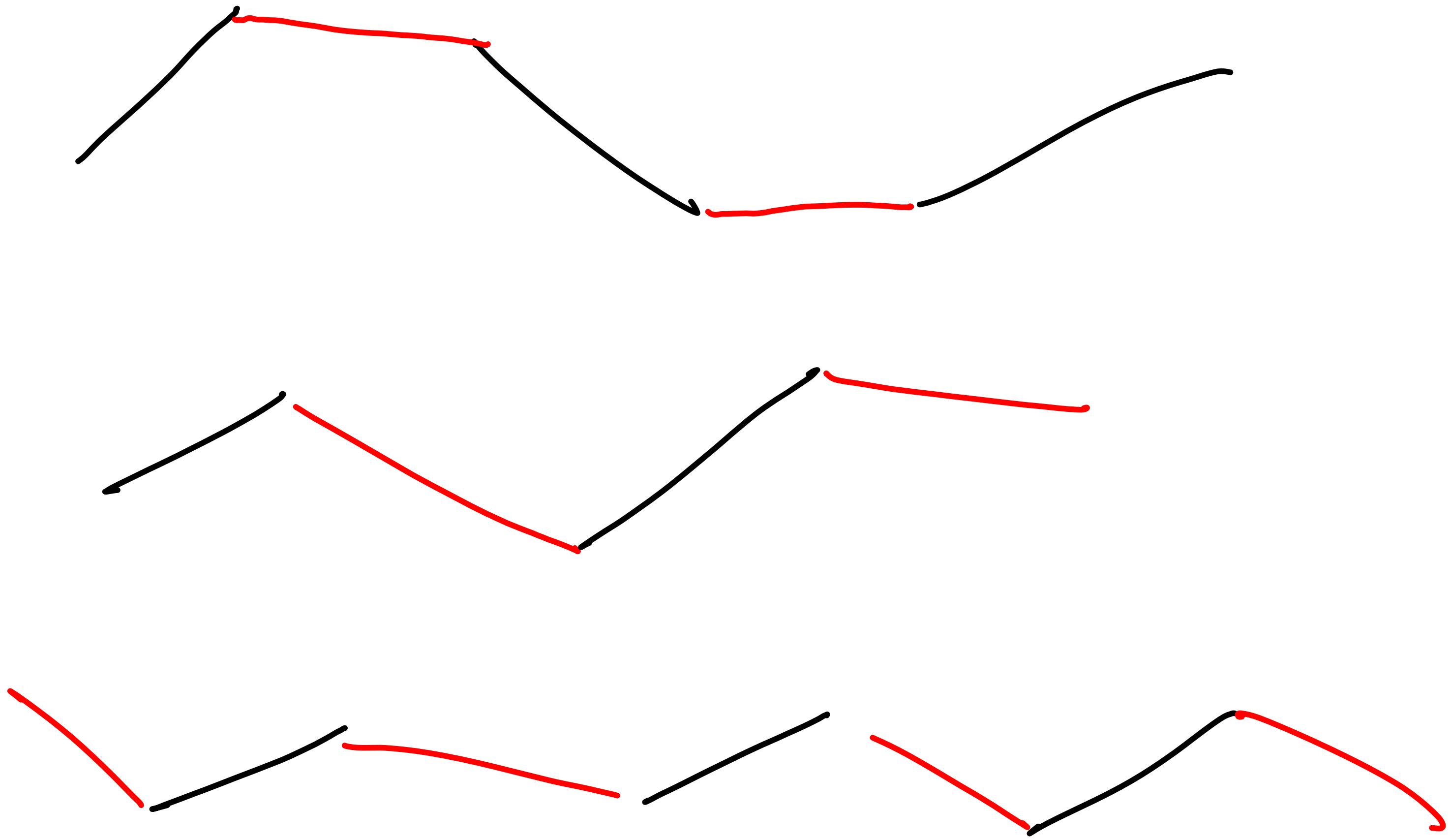


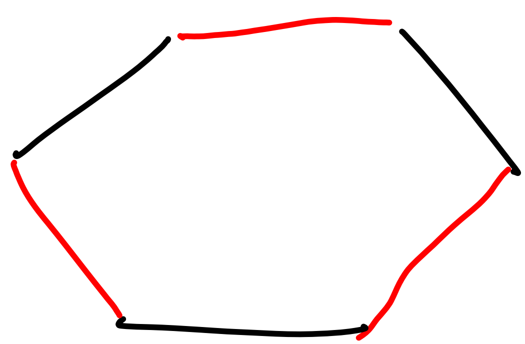
(ii) Suppose M is a matching and M' is a larger matching.

Consider

$$M \oplus M' = (M \setminus M') \cup (M' \setminus M)$$

consists of alternating paths and cycles.

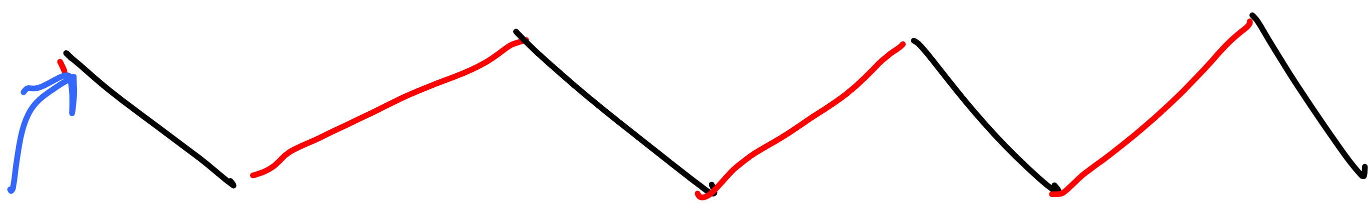




Every vertex has degree 0, 1 or 2 in $M \oplus M'$.

Only components that contain more M' than M edges

are



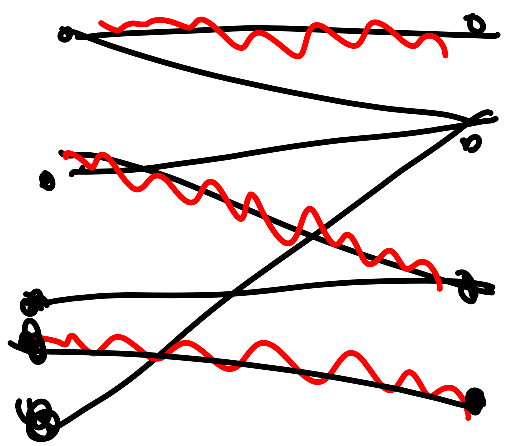
unsaturated by M

augmenting w.r.t. M

$|M'| > |M|$ and so there is one of these. $\square$

Bipartite Case

$G = (A \cup B, E)$



If M is a matching then

$|M| \le |A|, |B|$

When is there a matching of size $|A| \le |B|$.

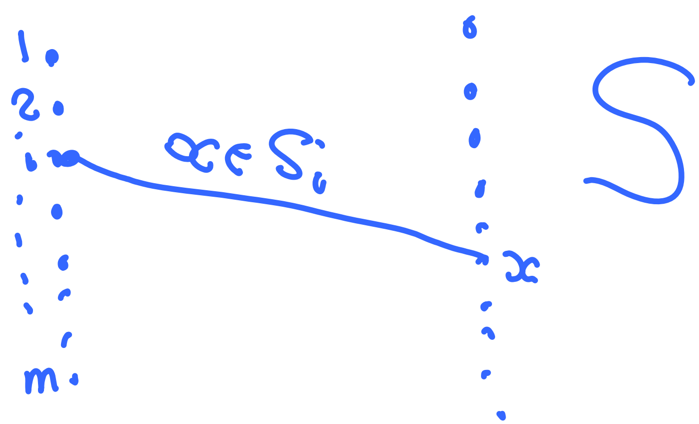
System of distinct representatives.

$S_1, S_2, \ldots, S_m$ are sets.

An S.D.R. is a set $\overbrace{\{s_1, s_2, \ldots, s_m\}}^{\text{distinct}}$

such that $s_i \in S_i$, $i = 1, 2, \cdots m$

Suppose $S = S_1 \cup S_2 \cup \cdots \cup S_m$



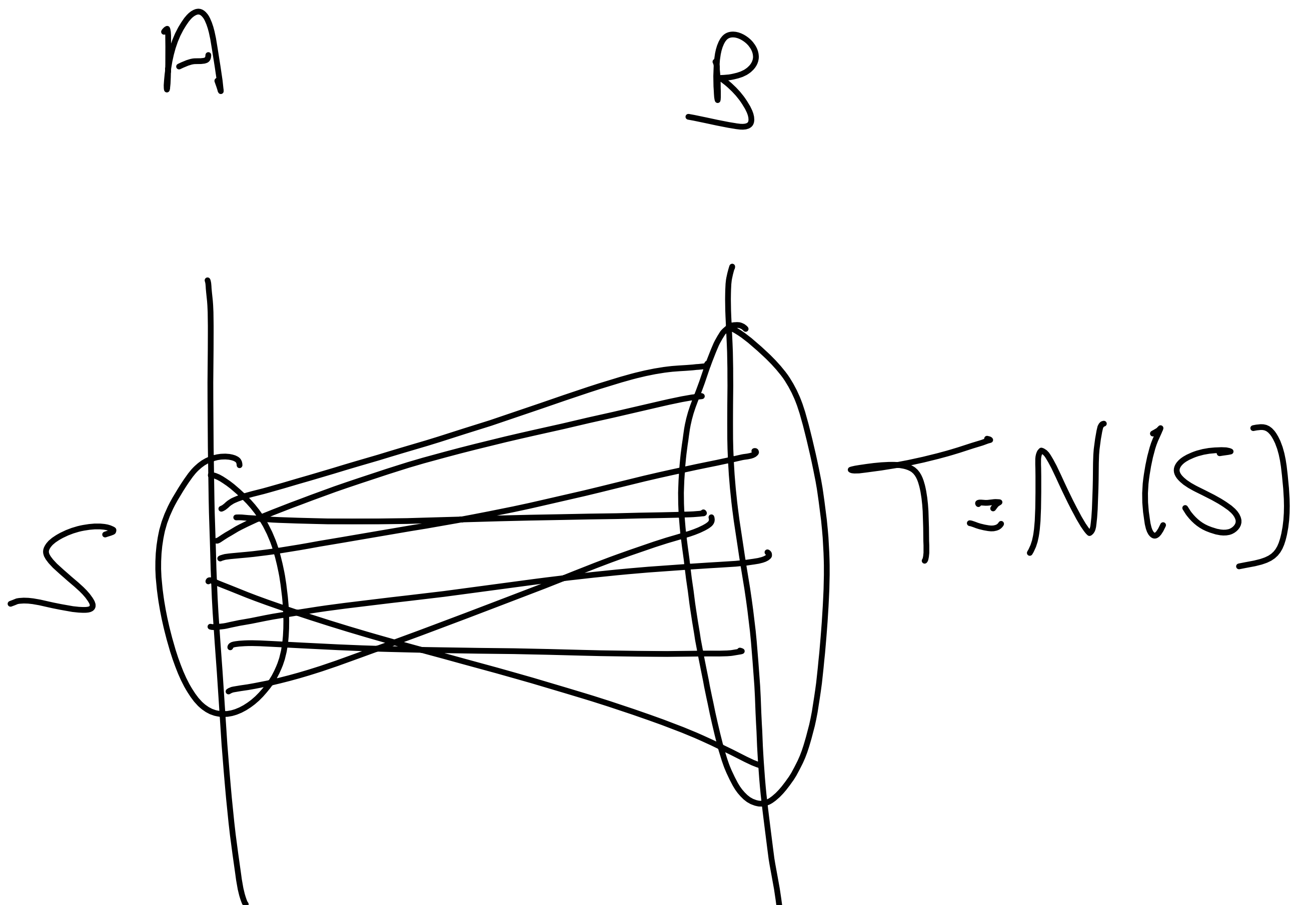
S.D.R. corr. to matching of left to right.

HALL's Theorem

G contains a matching of size $|A|$ iff for every $S \subseteq A$,

$$|N(S)| \geq |S|$$

{neighbors of vertices in S}



$$N(S) = \{ w \in B : \exists \; v \in S \text{ and } (v, w) \in E \}$$

(1) Suppose $\exists S$ s.t. $|N(S)| < |S|$