10\1\10

First moment method:

Let X be a random variable taking values in $\{0, 1, 2, \ldots\}$

$$\Pr[X \geq 1] \leq E[X].$$

$$E(X) = E(X \mid X=0)\Pr[X=0] + E(X \mid X \geq 1)\Pr(X \geq 1)$$

$$\geq 0 + 1 \cdot \Pr(X \geq 1).$$

Union distinct families

$\mathcal{A}$ is a family of sub-sets of $[n]$.

$\mathcal{A}$ is union-free if for distinct $A, B, C, D \in \mathcal{A}$ we have

$$A \cup B \neq C \cup D$$

We can get a large union free family by choosing randomly.

Choose $X_1, X_2, \ldots, X_p$ where X_i is a random subset of $[n]$.

Z = # of "bad" 4-tuples $\qquad A \cup B = C \cup D$

$$\Pr(Z \geq 1) \leq E(Z)$$

$$= \sum_{i,j,k,l} \Pr[X_i \cup X_j = X_k \cup X_l]$$

$$= p(p-1)(p-2)(p-3) \Pr[X_1 \cup X_2 = X_3 \cup X_4]$$

$$\Pr(Z \geq 1) \leq E(Z)$$

$$= \sum_{i,j,k,l} \Pr[X_i \cup X_j = X_k \cup X_l]$$

$$= p(p-1)(p-2)(p-3)\,\Pr[X_1 \cup X_2 = X_3 \cup X_4]$$

Think of a subset as a sequence of 0's & 1's.

$$\begin{array}{ll}
\begin{matrix} 1 \\ \end{matrix} & \\
[\,a_1 \quad \cdots\,] & X_1 \\
[\,b_1 \quad \cdots\,] & X_2 \\
[\,c_1 \quad \cdots\,] & X_3 \\
[\,d_1 \quad \cdots\,] & X_4
\end{array}$$

$\frac{3}{16}$ chance $a_1 + b_1 = 1$

& $c_1 + d_1 = 0$

$\vdots$

$$\Pr[\max\{a_1, b_1\} = \max\{c_1, d_1\}] = 5/8$$

$$E(Z) < p^4 \left(\frac{5}{8}\right)^n$$

So if $p = \left(\frac{8}{5}\right)^{n/4}$ then

$$\Pr(Z \geq 1) < \underline{\quad \quad \quad}$$

Similar problem: $X_1, X_2, \ldots X_p$

Person i know keys X_i

i.e. suppose $X_j = \{3, 5, 8, \ldots\}$

j knows key $3, 5, 8, \ldots$

n keys

If i wants to communicate with j.

How does i convince j that it is i

i sends key $X_i \cap X_j$ to j

This works if $\exists\, a, b, c$ s.t. $X_a \cap X_b \subseteq X_c$

Average Case of Quicksort

Distinct $x_1, \ldots, x_n$

(i) Choose $p \in [n]$ at random.

(ii) Divide remaining numbers into

$L = \{ j : x_j < x_p \}$ & $R = \{ j : x_j > x_p \}$

(iii) Apply Quicksort to L, R

$$T_n = E(\#\text{ comparisons})$$

$$= \sum_{i=1}^{n} E(\#\text{ comparisons} \mid p \text{ is the } i\text{th largest}) \times P_i(p \text{ is } i\text{th largest})$$

$$= \sum_{i=1}^{n} \left[\underbrace{n-1}_{\text{to construct } L,\,R} + T_{i-1} + T_{n-i}\right] \times \frac{1}{n}$$