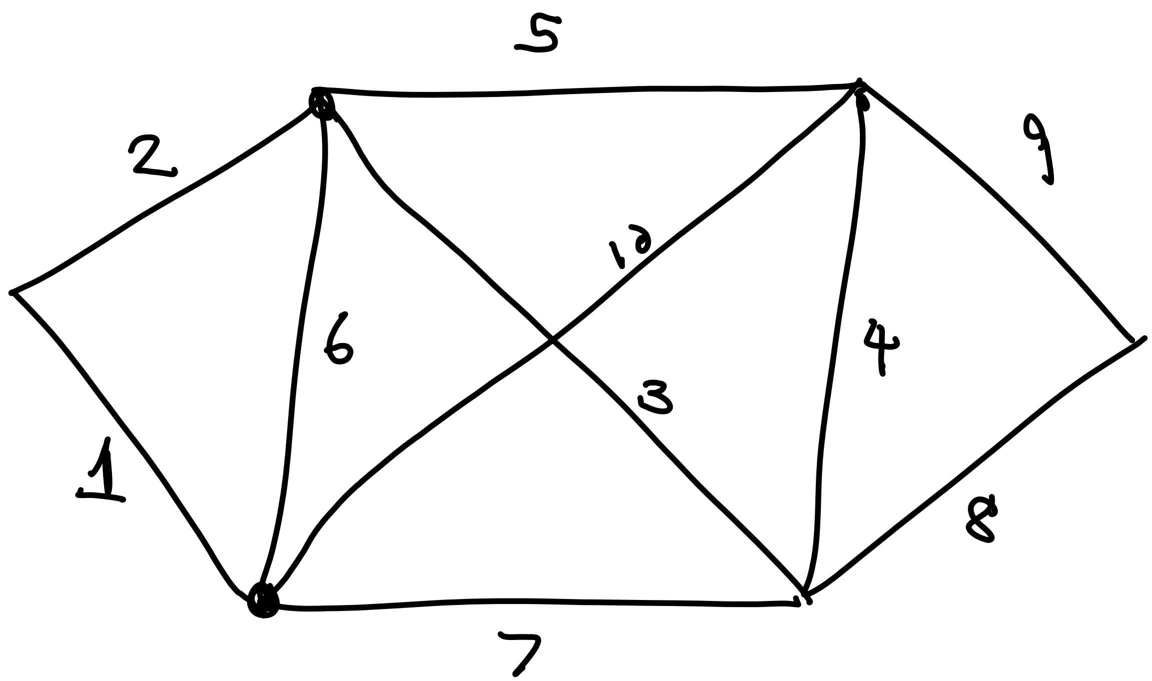
10\11\10

Eulerian Graphs

An Eulerian Cycle of graph G is closed walk that goes through each edge exactly once.



Thm

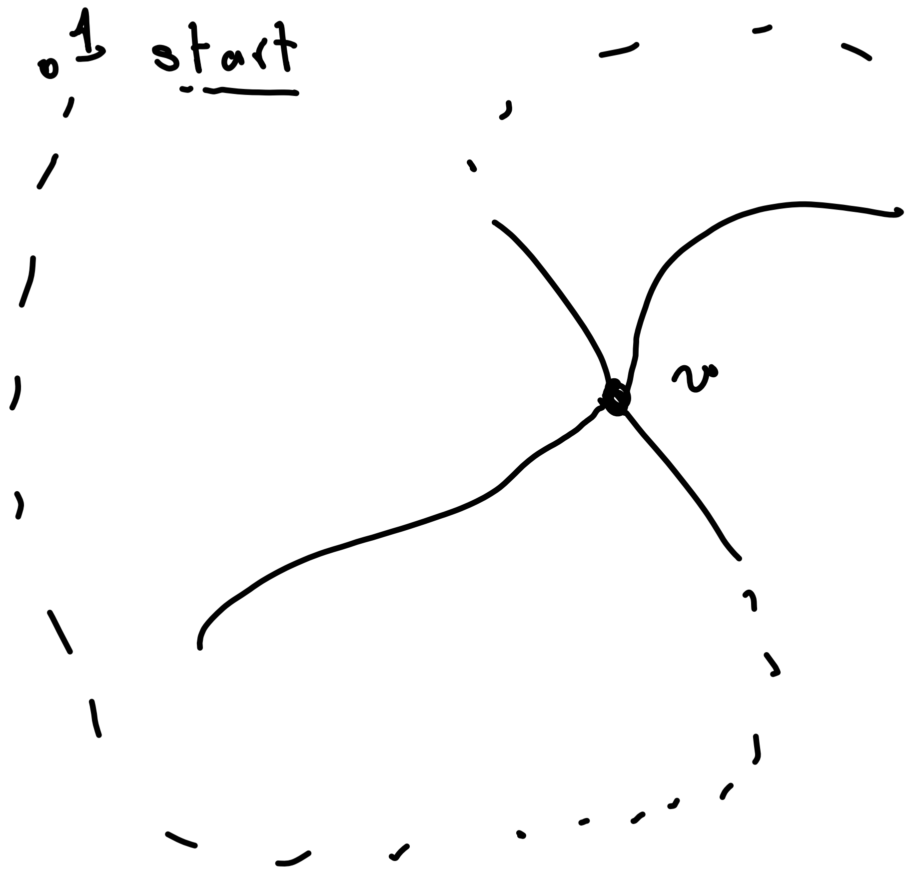
A graph is Eulerian iff

(i) It is connected

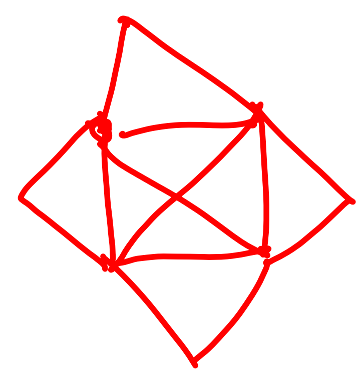
(ii) Every vertex has even degree.

Only If

Suppose G is Eulerian



For every incoming edge, there's an outgoing edge

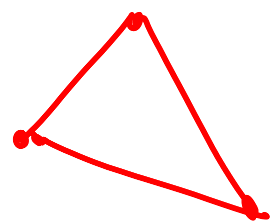


If

Assume G is connected and every vertex has even degree.

Proof by induction on $|E| + |V|$.

Base Case:



Inductive Step

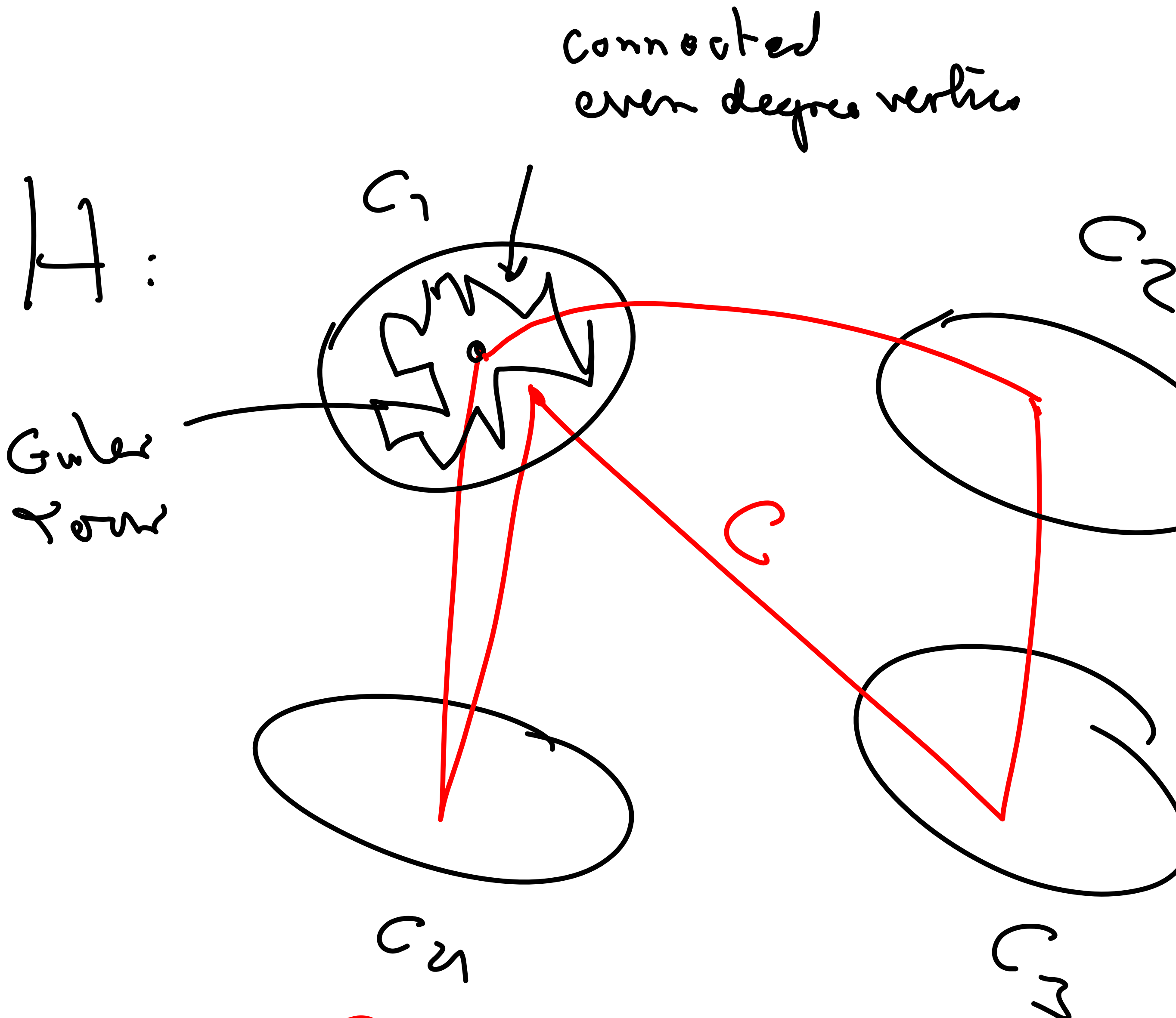
G is connected

It is not a tree. It has no vertex of

degree 1.

It contains a cycle C

$H = G - (\text{edges of } C)$



Go round C and when visiting C_i first time, do a tour.

Independent Sets & Cliques

$S \subseteq V$ is <u>independent</u> if it has no edges

$\alpha(G)$ = size of largest independent set.

Turán's Theorem

G has n vertices & m edges

$$\Rightarrow \quad \alpha(G) \geq \frac{n}{\frac{2m}{n} + 1}$$

average degree

Easwer theorem

$$\alpha(G) \geq \frac{n}{\Delta(G) + 1}$$

maximum degree

Greedy Algorithm

$I = \emptyset$; $S = V$

repeat

  choose $v \in S$;

  $I \leftarrow I + v$;

  $S \leftarrow S - N(v)$;

until $S = \emptyset$

output I

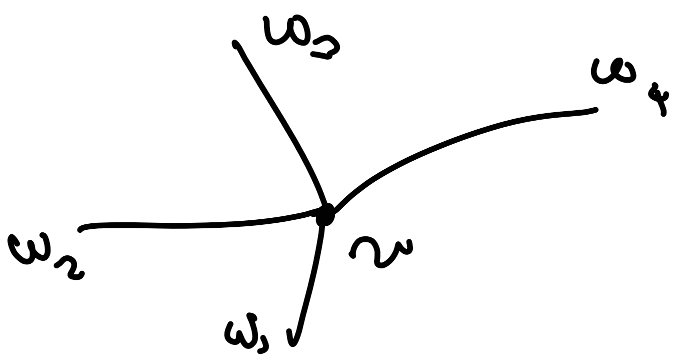
Each round removes $\leq$ $\Delta+1$ vertices

Proof Turán's Theorem

Let $\pi(1), \pi(2), \ldots, \pi(n)$

be a permutation of the vertex set of G (assuming $V = \{1, 2, \ldots, n\}$

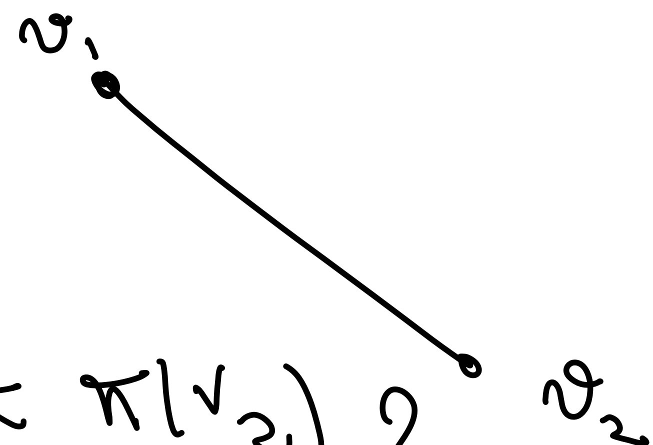
$$I(\pi) = \{v : \pi(w) > \pi(v) \ \forall w \in N(v)\}$$



keep v if

$\pi(v) < \pi(w_1),$
$\pi(w_2), \ldots$

$I(\pi)$ is an independent set



$\pi(v_1) < \pi(v_2)$?

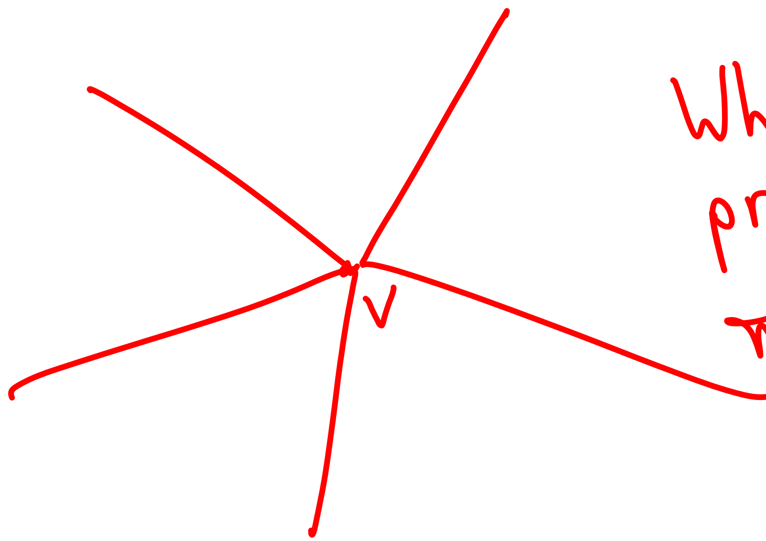
$\pi(v_2) < \pi(v_1)$,

Suppose π is random

$$E(|I(\pi)|) = ?$$

$$I(\pi) = \sum_{v \in V} \delta(v) \longleftarrow \begin{cases} 1 & v \in I \\ 0 & v \notin I \end{cases}$$

$$E(\delta(v)) = \frac{1}{d(v) + 1}$$



What is the probability that $\pi(v)$ is the minimum among v and its nbrs?

$\frac{1}{H} \geq \frac{1}{A}$

Therefore

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}$$

$\Rightarrow$ Turan's Theorem

Arithmetic Mean $\quad \dfrac{x_1 + x_2 + \cdots + x_m}{m} \quad A$

Geometric Mean $\quad (x_1 x_2 \cdots x_m)^{1/m} \quad G$

Harmonic Mean $\quad \dfrac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_m}} \quad H$

$G \leq \boxed{H \leq A}$

Computer Science Application

n processors

n values $x_1, x_2, \ldots, x_n$

In a round, each processor takes two values x_i & x_j and compares them and says which is the larger.

Then there is some 'discussion' among the processors

Question?

Is there an algorithm that finds maximum in constant number of rounds, $\forall$ sets of values.

NO - need $\Omega(\log\log n)$ rounds

First round of algorithm:

Choose n pairs of indices (i_k, j_k) $k = 1 \dots n$ and compare them.

$$G = \left([n], \{(i_1, j_1), (i_2, j_2) \dots (i_n, j_n)\}\right)$$